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Etude de la stabilisation exponentielle et polynomiale de certains systèmes d'équations couplées par des contrôles indirects bornés ou non bornés

JURY

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Contents

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Avant Propos

L'évolution au cours du temps de nombreux phénomènes physiques, biologiques, économiques ou mécaniques sont modélisés par des équations aux dérivées partielles (EDP) et/ou ordinaires (EDO).

Dans le cas du contrôle et de stabilisation des EDP, qui constitue le cadre de cette thèse, les modèles étudiés prennent en compte les variations temporelles et spatiales des variables qui traduisent l'état du système; ces problèmes se posent alors dans le cadre des systèmes dynamiques de dimension infinie. En pratique, du laboratoire de recherche jusqu'à la chaîne de production, pour étudier par exemple les moyens de limiter par auto-régulation les déformations de matériaux élastiques, ou d'agir extérieurement sur ces matériaux pour les ramener vers des états cibles souhaités, la question de la réponse d'un système dynamique à une action extérieure, ou à une action auto-régulante (appelée communément feedback) est essentielle. L'objectif est d'étudier la stabilisation de différents modèles de déformations de matériaux élastiques ou thermique. La plupart de ces modèles couplent des équations hyperboliques du second ordre. On s'intéressera particulièrement à la question de la stabilisation indirects de tels systèmes. Dans ce cas, l'action extérieure où l'action d'auto-régulation ne sont actives que sur certaines composantes du vecteur d'état. On souhaite alors savoir si cette action partielle directe est suffisante pour stabiliser l'ensemble des variables d'état.

Résumé de la thèse

La thèse porte essentiellement sur la stabilisation indirecte de certains système d'équations couplées moyennant un seul contrôle agissant localement à l'intérieur ou sur le bord du domaine. La nature du système ainsi couplé dépend du couplage des équations et du type de l'amortissement, et ceci donne divers résultats de stabilisation (exponentielle ou polynômiale) des systèmes étudiés.

D'abord, dans le cas de la stabilisation d'un système de Bresse formé de trois équations d'ondes couplées, un amortissement local de type chaleur est appliqué à une seule équation. Par une méthode fréquentielle combinée avec une méthode de multiplicateurs par morceau la décroissance exponentielle de l'énergie du système est établie sous la condition d'égalité de vitesses de propagation des ondes. Dans le cas contraire, une décroissance polynomiale est assurée.

Ensuite, un système de deux équations d'ondes couplées sous l'effet d'un seul amortissement frontière appliqué à une seule équation est considéré. Dans ce cas, la stabilité du système est influencée par la nature algébrique du terme de couplage ainsi que par la nature arithmétique de la quotient de vitesses de propagation des ondes. Par conséquence, différents résultats de stabilité exponentielle ou

polynomiale sont établis. Une étude spéctrale conduit à l'optimalié des résultats obtenus.

Finalement, dans le cas de la stabilisation d'un système de deux équations d'ondes couplées, un amortissement localement distribué de type Kelvin-Voight est appliqué à une seule equation. D'abord, d'après un théorème de Hormander, un résultat d'unicité est montré et par conséquent la stabilité forte du système est assurée. Ensuite, une décroissance polynomiale de l'énergie du système est établie.

Introduction

Control theory can be described as the process of influencing the behavior of a physical system to achieve a desired goal, primarily through the use of feedback which monitors the effect of a system and modifies its output. It is applied in a diverse range of scientific and engineering disciplines such as the reduction of noise, the vibration of structures like seismic waves and earthquakes, the regulation of biological systems like human cardiovascular system, the design of robotic systems, laser control in quantum mechanical and molecular systems.

In this thesis, we implement the semigroup theory in the spirit of spectral theory to study the approximations and stabilization of some coupled equations. In general, stability results are obtained using different methods like the spectral decomposition theory, the frequency domain approach combined with a piece-wise multipliers method.

0.1 Outline of the thesis

This thesis is divided into four main chapters. In the first chapter, we recall some basic definitions and theorems about the semigroup and spectral analysis theories.

Chapter two, as in [25], is devoted to the study of the energy decay rate of the Bresse system with one locally thermal dissipation law. We study the energy decay rate of the following weakly locally damped thermoelastic Bresse system:

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi + l\omega)_x - \kappa_0 l(\omega_x - l\varphi) = 0 \quad \text{in } (0, L) \times (0, \infty), \tag{0.1.1}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi + l\omega) + \alpha(x)\theta_x = 0 \quad \text{in } (0, L) \times (0, \infty), \tag{0.1.2}$$

$$\rho_1\omega_{tt} - \kappa_0(\omega_x - l\varphi)_x + \kappa l(\varphi_x + \psi + l\omega) = 0 \quad \text{in } (0, L) \times (0, \infty), \tag{0.1.3}$$

$$\rho_3\theta_t - \theta_{xx} + T_0(\alpha\psi_t)_x = 0 \quad \text{in } (0, L) \times (0, \infty). \tag{0.1.4}$$

With one of the following boundary conditions

$$\varphi(x,t) = \psi_x(x,t) = \omega_x(x,t) = \theta(x,t) = 0 \quad \text{for } x = 0, L, \tag{0.1.5}$$

$$\varphi(x,t) = \psi(x,t) = \omega(x,t) = \theta(x,t) = 0 \quad \text{for } x = 0, L, \tag{0.1.6}$$

and initial conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x),\ \ \psi(x,0) = \psi_0(x),\ \ \psi_t(x,0) = \psi_1(x) \\ \omega(x,0) = \omega_0(x),\ \ \omega_t(x,0) = \omega_1(x),\ \ \theta(x,0) = \theta_0(x) \end{cases} \tag{0.1.7}$$

where φ, ψ, ω are the vertical, shear angle and longitudinal displacements; θ is the temperature deviations from the reference temperature T_0 along the shear angle displacement. Here $\rho_1 = \rho A$, $\rho_2 = \rho I$, $\rho_3 = \rho c$, $\kappa_0 = EA$, $\kappa = \kappa' GA$, $b = EI$ and $l = R^{-1}$ are positive constants for the elastic and thermal material properties. To be more precise, ρ for density, E for the modulus of elasticity, G for the shear modulus, κ' for the shear factor, A for the cross-sectional area, I for the second moment of area of cross-section, R for the radius of the curvature and c for the thermal material property (for more details see Lagnese et al. [16]). The velocities of waves propagations are, respectively, $v_1 = \frac{\kappa}{\rho_1}$, $v_2 = \frac{b}{\rho_2}$, $v_3 = \frac{\kappa_0}{\rho_1}$.
The energy of solutions of the system (0.1.1)-(0.1.4) subject to initial state (0.1.7)

to either the boundary conditions (0.1.5) or (0.1.6) is defined by

$$E(t) = \frac{1}{2}\int_0^L \{\kappa|\psi + \varphi_x + l\omega|^2 + b|\psi_x|^2 + \kappa_0|\omega_x - l\varphi|^2 + \rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + \rho_1|\omega_t|^2 + \frac{\rho_3}{T_0}|\theta|^2\}dx. \tag{0.1.8}$$

then a straightforward computation gives :

$$\frac{d}{dt}E(t) = -\frac{1}{T_0}\int_0^L |\theta_x|^2 dx \leq 0. \tag{0.1.9}$$

Then the thermoelastic Bresse system is dissipative in the sense that its energy is non increasing with respect to the time t. Our goal is to study the effect of this dissipation on the Bresse system.

Different types of damping have been introduced to the Bresse system and several uniform and polynomial stability results have been obtained. We start by recalling some results related to the stabilization of the elastic Bresse system. Wehbe and Youssef [34], considered the elastic Bresse system subject to two locally internal dissipation laws. They proved that the system is exponentially stable if and only if the wave propagation speeds are equal. Otherwise, only a polynomial stability holds. Alabau-Boussouira et al. [2], considered the same system with one globally distributed dissipation law. The authors proved that, in general, the system is not exponentially stable but there exists polynomial decay with rates that depend on some particular relation between the coefficients. Using boundary conditions of Dirichlet-Dirichlet-Dirichlet type, they proved that the energy of the system decays at a rate $t^{-\frac{1}{3}}$ and at the rate $t^{-\frac{2}{3}}$ if $\kappa = \kappa_0$. These results are completed by Fatori and Montiero [12]. Using boundary conditions of Dirichlet-Neumann-Neumann type, the authors showed that the energy of the elastic Bresse system decays polynomially at the rate $t^{-\frac{1}{2}}$ and at the rate t^{-1}if $\kappa = \kappa_0$. Noun and Wehbe [26] extended the results of [2] and [12]. The authors considered the elastic Bresse system subject to one locally distributed feedback with Dirichlet-Neumann-Neumann or Dirichlet-Dirichlet-Dirichlet boundary conditions type. They proved that the exponential decay rate is preserved when the wave propagation speeds are equal. On the contrary, the authors established a polynomial energy decay with rates that depend on some particular relation between the coefficients and they obtained the rate of $t^{-\frac{1}{2}}$ or t^{-1}. Finally, see [32] for the stabilization of the elastic Bresse system with internal indefinite damping and [17] for the stabilization of elastic Bresse system with a nonlinear damping acting in the equation of the shear angle displacement, and nonlinear localized damping in other equations.

For the thermoelastic Bresse system, subject of this paper, there exist two important results. The first result is due to Liu and Rao [22], when they considered the Bresse system with two thermal dissipation laws. The authors showed that the energy decays exponentially when the wave speed of the vertical displacement coincides with the wave speed of longitudinal displacement or of the shear angle displacement. Otherwise, they found polynomial decay rates depending on the boundary conditions. When the system is subject to Dirichlet-Neumann-Neumann boundary conditions, they showed that the energy decays at the rate $t^{-\frac{1}{2}}$ and for fully Dirichlet boundary conditions, they proved that the energy of the system decays as $t^{-\frac{1}{4}}$. This result has been recently improved by Fatori and Rivera [13] in the sense that the authors considered only one globally dissipative mechanism given by one temperature, and they established the rate of decay $t^{-\frac{1}{3}}$ for Dirichlet-Neumann- Neumann and Dirichlet-Dirichlet-Dirichlet boundary conditions type. The main result of this paper is to extend the results from [13], by taking into consideration the important case when the thermal dissipation law is locally distributed on the angle displacement equation i.e the damping coefficient α is not constant but it is a positive function in $W^{2,\infty}(0,L)$ and strictly positive in an open subinterval $]a,b[\subset]0,L[$ (the cases $a=0$ or $b=L$ are not excluded) and to improve the polynomial energy decay rate.

In this chapter, we consider the Bresse system damped by one thermal dissipation law acting locally on the angle displacement equation with Dirichlet-Neumann-Neumann or Dirichlet-Dirichlet-Dirichlet boundary conditions types. Following the two types of boundary conditions, we define the energy spaces

$$\mathcal{H}_1 = H_0^1 \times (H_*^1)^2 \times (L^2)^2 \times L_*^2 \times L^2 \text{ and } \mathcal{H}_2 = (H_0^1)^3 \times (L^2)^4,$$

where

$$L_*^2 = \{f \in L^2(0,L) : \int_0^L f(x)dx = 0\} \text{ and } H_*^1 = \{f \in H^1(0,L) : \int_0^L f(x)dx = 0\}.$$

Both spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are equipped with the inner product which induces the energy norm

$$\|U\|_{\mathcal{H}_j}^2 = \kappa\|\varphi_x + \psi + l\omega\|^2 + b\|\psi_x\|^2 + \kappa_0\|\omega_x - l\varphi\|^2 \\ +\rho_1\|u\|^2 + \rho_2\|v\|^2 + \rho_1\|z\|^2 + \frac{\rho_3}{T_0}\|\theta\|^2. \quad (0.1.10)$$

Here and after, $\|.\|$ denotes the $L^2(0,L)$ norm. Moreover, we rewrite system (0.1.1)-(0.1.4) into an abstract form

$$U_t = \mathcal{A}_j U, \quad U(0) = U_0 \quad (0.1.11)$$

where $\mathcal{A}_j$, $j = 1, 2$ is a unbounded linear m-dissipative operator in the energy space $\mathcal{H}_j$. Consequently, system (0.1.1)-(0.1.4) is well-posed in the sense of semigroup of contraction (see [27]). In addition, since the resolvent of $\mathcal{A}$ is compact in the energy space $\mathcal{H}$, then using the spectral decomposition theory of Benchimol [7] (see also [5], we deduce that the Bresse system is strongly stable in the sense that

$$\lim_{t\to+\infty} E(t) = \frac{1}{2}\lim_{t\to+\infty} \|e^{tA_j}U_0\|_{\mathcal{H}_j} = 0 \quad j = 1,\, 2 \quad (0.1.12)$$

for all $U_0 \in \mathcal{H}_j$. Next, under the equal speed wave propagation condition, $\kappa = \kappa_0$ and $\frac{\rho_1}{\rho_2} = \frac{\kappa}{b}$, using a frequency domain approach combining with a piecewise multiplier method, we establish the following energy estimate:

$$E(t) = \frac{1}{2}\|e^{t\mathcal{A}_j}U_0\|_{\mathcal{H}_j} \leq Me^{-\epsilon t}\|U_0\|_{\mathcal{H}_j},\, t \geq 0 \quad \forall U_0 \in \mathcal{H}_j, \quad j = 1,\, 2 \quad (0.1.13)$$

where $M \geq 1$ and $\epsilon > 0$ are constants independent of U_0. Finally, in the natural general case, when $\kappa \neq \kappa_0$ and $\frac{\rho_1}{\rho_2} \neq \frac{\kappa}{b}$, we establish a new polynomial energy decay rate

$$E(t) \leq C\frac{1}{\sqrt{t}}\|U_0\|^2_{D(\mathcal{A}_j)} \quad \forall t > 0 \tag{0.1.14}$$

for smooth solution. In particular, if $\kappa = \kappa_0$ and $\frac{\rho_1}{\rho_2} \neq \frac{\kappa}{b}$, we establish a new polynomial energy decay rate

$$E(t) \leq C\frac{1}{t}\|U_0\|^2_{D(\mathcal{A}_j)} \quad \forall t > 0 \tag{0.1.15}$$

for the smooth solution.

In the third chapter, we move on to another subject which treats the stabilization of a system of coupled wave equations with one boundary damping. Let as recall that in [3], Ammar-Khodja and Bader studied the simultaneous boundary stabilization of a system of two wave equations coupling through the velocity terms. The system is described by:

$$u_{tt} - u_{xx} + b(x)y_t = 0 \quad \text{in } (0,1)\times(0,\infty), \tag{0.1.16}$$

$$y_{tt} - ay_{xx} - b(x)u_t = 0 \quad \text{in } (0,1)\times(0,\infty), \tag{0.1.17}$$

$$y_t(0,t) - \alpha\left(y_x(0,t) + u_t(0,t)\right) = 0 \quad \text{in } (0,\infty), \tag{0.1.18}$$

$$au_x(0,t) - y_t(0,t) = 0 \quad \text{in } (0,\infty), \tag{0.1.19}$$

$$u(1,t) = y(1,t) = 0 \quad \text{in } (0,\infty), \tag{0.1.20}$$

where $a > 0$, $\alpha > 0$ are constants and $b \in C^0([0,1])$. Under the equal speed wave propagation condition $i.e.$ $a = 1$, the authors proved that, system (0.1.16)-(0.1.20) is uniformly stable if and only if it is strongly stable and the coupling parameter b verifies that $\bar{b} := \int_0^1 b(x)dx \neq (2k+1)\frac{\pi}{2}$ for any $k \in \mathbb{Z}$. Moreover, when $a \neq 1$, they proved that system (0.1.16)-(0.1.20) is uniformly stable if and only if it is strongly stable and there exist $p, q \in \mathbb{Z}$ such that $a = \frac{(2p+1)^2}{q^2}$. Noting that, the above system is directly damped by two related boundary controls. Moreover, in [33], Toufayli considered a multidimensional system of coupled wave equations subject to one boundary feedback. Under the equal speed wave propagation condition (in the case $a = 1$) and if the coupling parameter b is small enough, she established an exponential energy decay estimate. However, on the contrary, no stability type has been discussed. We think that the conditions on a and b are technical and could be improved. Then the influence of the arithmetic property of the ratio of the wave propagation speeds a and of the algebraic property of the coupling parameter b on the stability of the system of two coupled wave equations when only one of these equation is effectively damped remains an open problem. Our objective in this chapter is to give a complete answer of this interesting open problem on the one dimensional case.

Then, we consider a system of wave equations coupled by velocities with only one boundary damping. The system is described by:

$$\begin{aligned} u_{tt} - u_{xx} + by_t &= 0 \quad \text{in } (0,1) \times (0,\infty), && (0.1.21)\\ y_{tt} - ay_{xx} - bu_t &= 0 \quad \text{in } (0,1) \times (0,\infty), && (0.1.22)\\ y_x(0,t) - y_t(0,t) &= 0 \quad \text{in } (0,\infty), && (0.1.23)\\ u(1,t) = y(1,t) = u(0,t) &= 0 \quad \text{in } (0,\infty), && (0.1.24) \end{aligned}$$

where $a > 0$ and $b \in \mathbb{R}^{\star}$ are constants.

We define the space

$$H_R^1(0,1) = \left\{ y \in H^1(0,1) : y(1) = 0 \right\}$$

and the energy space

$$\mathcal{H} = H_0^1(0,1) \times L^2(0,1) \times H_R^1(0,1) \times L^2(0,1),$$

which is endowed with the inner product

$$\left(U,\widetilde{U}\right)_{\mathcal{H}} = \int \left(u_x\overline{\widetilde{u}_x} + v\overline{\widetilde{v}} + ay_x\overline{\widetilde{y}_x} + z\overline{\widetilde{z}}\right) dx \quad \forall\, U=(u,v,y,z), \widetilde{U}=(\widetilde{u},\widetilde{v},\widetilde{y},\widetilde{z}) \in \mathcal{H}.$$

We next define the unbounded linear operator $\mathcal{A}$: $D(\mathcal{A}) \subseteq \mathcal{H} \longrightarrow \mathcal{H}$, by

$$D(\mathcal{A}) = \left\{U=(u,v,y,z) \in \mathcal{H} : u,y \in H^2, v \in H_0^1, z \in H_R^1 \text{ and } y_x(0) = z(0)\right\} \tag{0.1.25}$$

and

$$\mathcal{A}U = (v, u_{xx} - bz, z, ay_{xx} + bv),\ \forall\ U=(u,v,y,z) \in D(\mathcal{A}). \tag{0.1.26}$$

If $U=(u,u_t,y,y_t)$ be a regular solution of system (0.1.21)-(0.1.24), then we rewrite this system as the following evolutionary equation

$$\begin{cases} U'(t) &= \mathcal{A}U(t), \\ U(0) &= U_0 \in \mathcal{H}. \end{cases} \tag{0.1.27}$$

It is easy to see that the operator $\mathcal{A}$ is m-dissipative on the energy space. Consequently, the system is well-posed in the sense of semigroup of contractions (see [27]). Our aim is to study the energy decay rate of system (0.1.21)-(0.1.24). First, we prove that system (0.1.21)-(0.1.24) is strongly stable if and only if

$$b^2 \neq \frac{4(k_1^2 - ak_2^2)(ak_1^2 - k_2^2)\pi^2}{(a+1)(k_1^2+k_2^2)}, \quad \forall k_1, k_2 \in \mathbb{Z}. \tag{SC1}$$

Consequently, the strong stability does not hold in general. Next, if the coupling parameter verifying (SC1), we show that the energy decay rate of system (0.1.21)-(0.1.24) is greatly influenced by the nature of the coupling parameter b (an additional condition on b) and by the arithmetic property of the ratio of the wave propagation speeds a. Indeed, in the case of $a=1$ when the waves propagate at the same speed and if $b \notin \pi\mathbb{Z}$ we establish an exponential stability of system (0.1.21)-(0.1.24) for usual initial data:

$$E(t) \leq Me^{-\omega t}E(0), \qquad \forall t > 0$$

where $M \geq 1$ and $\omega > 0$ are constants independent of U_0. In addition, using a spectral approach, we prove that the condition $b \notin \pi\mathbb{Z}$ is optimal in the sense that if $a = 1$ and there exist $k \in \mathbb{Z}$ such that $b = k\pi$, system (0.1.21)-(0.1.24) lakes its exponential stability. In this case, we show that the following polynomial energy decay rate is optimal

$$E(t) \leq C\frac{1}{\sqrt{t}}\|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0, \ \forall U_0 \in D(\mathcal{A}) \tag{0.1.28}$$

where $C > 0$ is a constant independent of U_0. Finally, assume that $a \neq 1$ and b satisfies condition (SC1). If $a \in \mathbb{Q}$ and b small enough **or** $\sqrt{a} \in \mathbb{Q}$. Then there exists a constant $C > 0$ such that for every initial data $U_0 = (u_0, u_1, y_0, y_1) \in D(\mathcal{A})$, the energy of system (0.1.21)-(0.1.24) verify the following estimate:

$$E(t) \leq C\frac{1}{\sqrt{t}}\|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0. \tag{0.1.29}$$

The notion of indirect damping mechanisms has been introduced by Russell in [30], and since this time, it retains the attention of many authors. In particular, the boundary stabilization of the system of two wave equations coupled through the zero order terms has been studied with different approaches. In [1], Alabau-Boussouira studied the boundary indirect stabilization of a system of two second order evolution equations coupled through the zero order terms. The lack of uniform stability was proved by a compact perturbation argument and a polynomial energy decay rate of type $\frac{1}{\sqrt{t}}$ is obtained by a general integral inequality in the case where the waves propagate at the same speed and Ω is a star-shaped domain in $\mathbb{R}^N$, or in the case where the ratio of the wave propagation speeds of the two equations is equal $1/k^2$ with k being an integer and Ω is a cubic domain of $\mathbb{R}^3$. Liu and

Rao in [21] considered a system of two coupled wave equations with one boundary damping and they proved that the energy of the system decays at the rate $\frac{1}{t}$ for smooth initial data on a N-dimensional domain Ω with usual geometrical condition when the waves propagate at the same speed. On the contrary, under some arithmetic condition on the ratio of the wave propagation speeds of the two equations, they established a polynomial energy decay rate for smooth initial data on an one-dimensional domain. Ammari and Mehrenberger in [4], gave a characterization of the stability of a system of two evolution equations coupling through the velocity terms subject to one bounded viscous feedback damping. Note nevertheless that the above system does not enter in the framework of this paper.

Chapter four is devoted to the study of coupled wave equations weakly coupled and partially damped. Let $\Omega \in \mathbb{R}^N$ be a bounded open set with Lipschitz boundary Γ. We consider the following system of coupled wave equations with a viscoelastic damping around the boundary Γ:

$$\begin{cases} \varrho_1(x)u_{tt} - \operatorname{div}(a_1(x)\nabla u + b(x)\nabla u_t) + \alpha y = 0, & \text{in} \quad \Omega \times \mathbb{R}^+, \\ \varrho_2(x)y_{tt} - \operatorname{div}(a_2(x)\nabla y) + \alpha u = 0, & \text{in} \quad \Omega \times \mathbb{R}^+, \\ u = y = 0, & \text{on} \quad \Gamma \times \mathbb{R}^+, \\ u(0) = u_0,\ y(0) = y_0\ u_t(0) = u_1,\ y_t(0) = y_1, & \text{in} \quad \Omega, \end{cases} \tag{0.1.30}$$

where $\varrho_1(x) \geq \varrho_1 > 0$, $\varrho_2(x) \geq \varrho_2 > 0$, $a_1(x) \geq a_1^0 > 0$, $a_2(x) \geq a_2^0 > 0$, and $b(x) \geq 0$ for all $x \in \Omega$, the coupling parameter α is a real number.

Let $U = (u, u_t, y, y_t)$ a regular solution of system (0.1.30). Then, the total natural energy of the system is given by:

$$E(t) = 1/2 \int_\Omega (\varrho_1(x)|u_t|^2 + a_1(x) \mid \nabla u \mid^2 + \varrho_2(x)|y_t|^2 \\ + a_2(x) \mid \nabla y \mid^2 + \alpha u y) dx. \tag{0.1.31}$$

By a straightforward calculation we obtain that

$$E'(t) = -\int_\Omega b \mid \nabla u_t \mid^2 dx \leq 0.$$

That is the system (0.1.30) is dissipative in the sense that its energy is decreasing with respect to the time t.

Let $\alpha_0 = \min \left(\frac{a_1^0}{c_0^2}, \frac{a_2^0}{c_0^2} \right)$ where c_0 is the Poincaré constant. In what follows, we assume that α is a real number such that $|\alpha| < \alpha_0$. Here and after, assume that coefficient functions ϱ_1, ϱ_2, a_1, b, $a_2 \in L^\infty(\Omega)$.

For any $\gamma > 0$, we define the γ-neighborhood O_γ of the boundary Γ as follows

$$O_\gamma := \{x \in \Omega : \inf_{y \in \Gamma} |x - y| \leq \gamma, \}, \tag{0.1.32}$$

and assume that there exist two constants b_0 and γ such that

$$b(x) \geq b_0 > 0, \qquad \forall x \in O_\gamma. \tag{SC}$$

We start by formulating system (0.1.30) as an abstract Cauchy problem in an appropriate Hilbert space. First, define the energy space $\mathcal{H}$ by

$$\mathcal{H} = (H_0^1(\Omega) \times L(\Omega))^2 \tag{0.1.33}$$

endowed with the inner product:

$$(U, V)_\mathcal{H} = \int_\Omega (a_1(x) \nabla u \cdot \nabla \tilde{u} + a_2(x) \nabla y \cdot \nabla \tilde{y}) dx + \int_\Omega (\varrho_1 v \tilde{v} + \varrho_2 z \tilde{z}) dx + \int_\Omega \alpha (u \tilde{y} + y \tilde{u}) dx$$

for all $U = (u, v, y, z)$, $V = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z}) \in \mathcal{H}$.

Next, define the unbounded linear operator $\mathcal{A}$ by :

$$D(\mathcal{A}) = \{(u, v, y, z) \in \mathcal{H} : \mathrm{div}(\mathrm{a_2(x)} \nabla \mathrm{y}), \mathrm{div}(\mathrm{a_1(x)} \nabla \mathrm{u} + \mathrm{b(x)} \nabla \mathrm{v}) \in \mathrm{L}^2(\Omega), \text{ and } \mathrm{v}, \mathrm{z} \in \mathrm{H}_0^1(\Omega)\}$$

$$\mathcal{A}U = (v, \frac{1}{\varrho_1}(\mathrm{div}(\mathrm{a}_1(\mathrm{x}))\nabla \mathrm{u}+\mathrm{b}(\mathrm{x})\nabla \mathrm{v}) - \frac{\alpha}{\varrho_1}\mathrm{y}, \mathrm{z}, \frac{1}{\varrho_2}\mathrm{div}(\mathrm{a}_2(\mathrm{x})\nabla \mathrm{y}) - \frac{\alpha}{\varrho_2}\mathrm{u}), \ \forall \mathrm{U} = (\mathrm{u},\mathrm{v},\mathrm{y},\mathrm{z}) \in \mathrm{D}(\mathcal{A}).$$

If $U = (u, u_t, y, y_t)$ is a regular solution of system (0.1.30), then we rewrite this system as the following evolutionary equation:

$$U_t = \mathcal{A}U, \quad U(0) = U_0 \ \in \mathcal{H}. \tag{0.1.34}$$

It is easy to see that the operator $\mathcal{A}$ is m-dissipative on the energy space $\mathcal{H}$. Then system (0.1.30) is well-posed in the sense of semigroup of contractions (see [27]). Now, assume that (SC) holds and a_1, $a_2 \in C^{0,1}(\overline{\Omega})$.Then, using a unique continuation result (see [14]) we show that system (0.1.30) is strongly stable in the sense that

$$\lim_{t\to\infty} E(t) = \frac{1}{2}\lim_{t\to\infty}\|e^{t\mathcal{A}}U_0\|^2 = 0, \quad \forall U_0 \in \mathcal{H}.$$

Moreover, by observing the eigenvalues of the operator $\mathcal{A}$, we deduce that system (0.1.30) is not uniformly exponentially stable. So it is natural to hope a polynomial energy decay. Assume that

$$a_1, a_2, \rho_1, \rho_2, b \in C^{1,1}(\bar{\Omega}). \tag{H1}$$

Also, we assume the following supplementary conditions.

There exist two functions $q, \hat{q} \in C^1(\Omega, \mathbb{R}^N)$ and $0 < \alpha < \beta < \gamma$, such that

$$\partial_j q_k = \partial_k q_j, \quad \mathrm{div}(\mathrm{a}_2\rho_2 \mathrm{q}) \in \mathrm{C}^{0,1}\,(\Omega_\beta) \quad \text{and} \quad \mathrm{q} = 0 \ \text{ on } \ \mathrm{O}_\alpha, \tag{H2}$$

$$\partial_j \hat{q}_k = \partial_k \hat{q}_j, \quad \mathrm{div}(\mathrm{a}_1\rho_1 \hat{\mathrm{q}}) \in \mathrm{C}^{0,1}\,(\Omega_\beta) \quad \text{and} \quad \hat{\mathrm{q}} = 0 \ \text{ on } \ \mathrm{O}_\alpha, \tag{H3}$$

There exists a constant $\sigma_1 > 0$, such that

$$2a_2\partial q_j + (q_k\partial_j a_2 + q_j\partial_k a_2) + \left[\frac{a_2}{\rho_2}(q\nabla\rho_2 - q\nabla a_2)\right] I \geq \sigma_1 I, \quad \forall x \in \Omega_\beta. \tag{H4}$$

There exists a constant $\sigma_2 > 0$, such that

$$2a_1\partial \hat{q} + (\hat{q}_k\partial_j a_1 + \hat{q}_j\partial_k a_1) + \left[\frac{a_1}{\rho_1}(\hat{q}\nabla\rho_1 - \hat{q}\nabla a_1)\right] I \geq \sigma_2 I, \quad \forall x \in \Omega_\beta. \tag{H5}$$

There exists a constant $M > 0$ such that for all $v \in H_0^1(\Omega)$, we have

$$|(\hat{q} \cdot \nabla v)\nabla b - (\hat{q} \cdot \nabla b)\nabla v| \leq M\sqrt{b}\ |\nabla v|\,, \quad \forall x \in \Omega_\beta. \tag{H6}$$

Under conditions (SC), (H1)- (H6), using a frequency domain approach (see [9]) combining with piece-wise multiplier method, we deduce that there exists a constant $C > 0$ such that for every initial data $U_0 = (u_0, u_1, y_0, y_1) \in D(\mathcal{A})$, the energy of system (0.1.30) verify the following estimate:

$$E(t) \leq C\frac{1}{\sqrt[4]{t}}\|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0. \tag{0.1.35}$$

The local viscoelastic damping is a natural phenomena of bodies which have one part made of viscoelastic material, and the other is made of elastic material. There are a few number of publications concerning the wave equation with local viscoelastic damping. In [19], Liu and Rao studied the stability of a wave equations with local viscoelastic damping distributed around the boundary of the domain. They proved that the energy of the system goes exponentially to zero for all usual initial data. K. Liu and Z. Liu in [18], considered the longitudinal and transversal vibrations of the Euler-Bernoulli beam with Kelvin-Voigt damping distributed locally on any subinterval of the region occupied by the beam. They proved that the semigroup associated with the equation for the transversal motion of the beam is exponentially stable, although the semigroup associated with the equation for the longitudinal motion of the beam is not exponentially stable. Chen et. al in [11], studied the mathematical properties of a variational second order evolution equation, which includes the equations modelling vibrations of the Euler-Bernoulli and Rayleigh beams with the global or local Kelvin-Voigt damping.

Chapter 1

Preliminaries

As the analysis done in this Ph.D. thesis is based on the semigroup and spectral analysis theories, we recall, in this chapter, some basic definitions and theorems which will be used in the following chapters. We refer to [27, 23, 20, 6, 9, 15, 28].

1.1 Semigroups

Numerous physical models can be written in the form of an abstract Cauchy problem

$$\begin{cases} \dot{x}(t) &= \mathcal{A}x(t), \quad t > 0, \\ x(0) &= x_0, \end{cases} \tag{1.1.1}$$

where $(\dot{})$ denotes the derivative with respect to time t, $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup $T(t)$ over a Hilbert space $\mathcal{H}$ and $x_0 \in \mathcal{H}$ is given. We are looking for a solution $x : \mathbb{R}_+ \to \mathcal{H}$. Therefore, we start by introducing some basic concepts concerning the semigroups.

Definition 1.1.1. *Let X be a Banach space.*

1) A one parameter family $T(t)$, $t > 0$, of bounded linear operators from X into X is a semigroup of bounded linear operators on X if

(i) $T(0) = I$;

(ii) $T(t+s) = T(t)T(s)$ for every $s, t \geq 0$.

2) A semigroup of bounded linear operators, $T(t)$, is uniformly continuous if

$$\lim_{t \to 0^+} \|T(t) - I\|_{\mathcal{L}(\mathcal{H})} = 0.$$

3) A semigroup $T(t)$ of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators or a C_0 semigroup if

$$\lim_{t \to 0^+} T(t)x = x.$$

4) The linear operator $\mathcal{A}$ defined by

$$D(\mathcal{A}) = \left\{ x \in X;\ \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{ exists} \right\}$$

and

$$\mathcal{A}x = \lim_{t \to 0} \frac{T(t)x - x}{t}, \quad \forall x \in D(\mathcal{A}),$$

is the infinitesimal generator of the semigroup $T(t)$.

Theorem 1.1.2. *Let $T(t)$ be a C_0-semigroup. Then there exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\|_{\mathcal{L}(\mathcal{H})} \leq M e^{\omega t}, \quad \forall t > 0.$$

In the above theorem, if $\omega = 0$, then $T(t)$ is called uniformly bounded and if moreover $M = 1$, then $T(t)$ is called a C_0 semigroup of contractions.

Definition 1.1.3. *Let $\mathcal{H}$ be a Hilbert space. An operator $(\mathcal{A}, D(\mathcal{A}))$ on $\mathcal{H}$ satisfying*

$$\Re(\mathcal{A}U, U) \leq 0, \quad \forall U \in D(\mathcal{A})$$

is said to be a dissipative operator. A maximal dissipative operator $(\mathcal{A}, D(\mathcal{A}))$ on $\mathcal{H}$ is a dissipative operator for which $R(\lambda I - \mathcal{A}) = \mathcal{H}$, for some $\lambda > 0$. A maximal dissipative operator is also called m-dissipative operator.

For the existence of solutions, we normally use the following Lumer-Phillips Theorem or Hille-Yosida Theorem.

Theorem 1.1.4. *(Lumer-Phillips Theorem) Let $\mathcal{A}$ be a linear operator with dense domain $D(\mathcal{A})$ in a Banach space X.*

(i) If $\mathcal{A}$ is dissipative and there exists a $\lambda_0 > 0$ such that the range $\mathcal{R}(\lambda_0 I - \mathcal{A}) = X$, then $\mathcal{A}$ generates a C_0 semigroup of contractions on X.

(ii) If $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on X then $\mathcal{R}(\lambda I - \mathcal{A}) = X$ for all $\lambda > 0$ and $\mathcal{A}$ is dissipative.

Consequently, $\mathcal{A}$ is maximal dissipative on a Hilbert space $\mathcal{H}$ if and only if it generates a C_0- semigroup of contractions on $\mathcal{H}$ and thus the existence of the solution is justified by the following corollary which follows from Lumer-Phillips theorem.

Corollary 1.1.5. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{A}$ be a linear operator defined from $D(\mathcal{A}) \subset \mathcal{H}$ into $\mathcal{H}$. If $\mathcal{A}$ is maximal dissipative then the initial value problem* (1.1.1) *has a unique weak solution $x \in C([0, +\infty), \mathcal{H})$, for each initial data $x_0 \in \mathcal{H}$. Moreover, if $x_0 \in D(\mathcal{A})$, then $x \in C^0([0, +\infty), D(\mathcal{A})) \cap C^1([0, +\infty), \mathcal{H})$.*

1.2 Stability of semigroups

After recalling some results concerning the well posedness of system (1.1.1), we aim to discuss the type of stability of the solution. We introduce here the notions of stability that we encounter in this work.

Assume that $\mathcal{A}$ is a generator of a C_0-semigroup of contractions $e^{t\mathcal{A}}$ on a Hilbert space $\mathcal{H}$. We say that the semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is

i Strongly (asymptotically) stable if for all $x_0 \in \mathcal{H}$

$$\lim_{t\to+\infty} \|e^{t\mathcal{A}}x_0\|_{\mathcal{H}} = 0.$$

ii Exponentially (or uniformly) stable if there exist two positive constants C, ω such that

$$\|e^{t\mathcal{A}}x_0\|_{\mathcal{H}} \leq Ce^{-\omega t}\|x_0\|_{\mathcal{H}}, \quad \forall t > 0, \forall x_0 \in \mathcal{H}.$$

iii Polynomially stable if there exist constants α, β, $C > 0$ such that

$$\|e^{t\mathcal{A}}(d - \mathcal{A})^{-\alpha}\| \leq Ct^{-\beta}, \quad \forall t > 0$$

for some $d > 0$.

Now we recall a result in [15, 28] which gives necessary and sufficient conditions for which a semigroup is exponentially stable.

Theorem 1.2.1. *Let $T(t)$ be a C_0 semigroup on a Hilbert space $\mathcal{H}$ and $\mathcal{A}$ be its infinitesimal generator. $T(t)$ is exponentially stable; i.e., there exists M and α positive constants such that $\|T(t)\| \leq Me^{-\alpha t}$ if and only if*

(i) $i\mathbb{R} \subseteq \rho(\mathcal{A})$, where $\rho(\mathcal{A})$ denotes the resolvent set of $\mathcal{A}$

and

(ii) $\sup_{\omega \in \mathbb{R}} \|(i\omega - \mathcal{A})^{-1}\| < \infty.$

When the exponential stability is attained, we search for the optimal exponential decay rate; mainly for the spectrum determined growth condition.

Definition 1.2.2. *Let $\mathcal{A}$ be the infinitesimal generator of a C_0 semigroup, $T(t)$, on a Hilbert space $\mathcal{H}$. Consider*

$$\omega(\mathcal{A}) := \inf\{\alpha \in \mathbb{R};\ \|T(t)\| \leq Me^{\alpha t}\} = \lim_{t \to \infty} \frac{1}{t} \log \|T(t)\|,$$

the growth exponent bound of $T(t)$, and

$$\mu(\mathcal{A}) = \sup\{\Re\lambda;\ \lambda \in \sigma(\mathcal{A})\},$$

the spectral abscissa of the operator $\mathcal{A}$ where $\sigma(\mathcal{A})$ denotes its spectrum. If $\omega(\mathcal{A}) = \mu(\mathcal{A})$, then we say that the spectrum determined growth condition holds.

Remark 1.2.3. *From the Hille-Yosida Theorem, we know that $\mu(\mathcal{A}) \leq \omega(\mathcal{A})$ for any infinitesimal generator of a strongly continuous semigroup. However, in general, $\omega(\mathcal{A}) \leq \mu(\mathcal{A})$ is not always true.*

If the semigroup fails to be exponentially stable, we search for another type of decay rate like the polynomial stability which is characterized by the following Theorem in [9].

Theorem 1.2.4. *Let $(T(t))_{t\geq 0}$ be a bounded C_0 semigroup on a Hilbert space $\mathcal{H}$ with a generator $\mathcal{A}$ such that $i\mathbb{R} \subseteq \rho(\mathcal{A})$. Then for a fixed $\alpha > 0$, the following conditions are equivalent:*

(i)

$$\|(is - \mathcal{A})^{-1}\| = O(|s|^{\alpha}), \quad s \to \infty;$$

(ii)

$$\|T(t)\mathcal{A}^{-\alpha}\| = O(t^{-1}), \quad t \to \infty;$$

(iii)

$$\|T(t)\mathcal{A}^{-1}\| = O(t^{\frac{-1}{\alpha}}), \quad t \to \infty.$$

Note that the notation $A = O(B)$ *means that there exists* $c > 0$ *such that* $|A| \leq c|B|$.

Chapter 2

Weakly locally thermal stabilization of Bresse system

2.1 Introduction and statement of the main result

In this chapter, we study the energy decay rate of the Bresse system subject to one locally temperature dissipation law operating on the angle displacement equation. The system is governed by the following partial differential equations:

$$\begin{aligned}
\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi + l\omega)_x - \kappa_0 l(\omega_x - l\varphi) &= 0 \quad \text{in } (0,L)\times(0,\infty), && (2.1.1)\\
\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi + l\omega) + \alpha(x)\theta_x &= 0 \quad \text{in } (0,L)\times(0,\infty), && (2.1.2)\\
\rho_1\omega_{tt} - \kappa_0(\omega_x - l\varphi)_x + \kappa l(\varphi_x + \psi + l\omega) &= 0 \quad \text{in } (0,L)\times(0,\infty), && (2.1.3)\\
\rho_3\theta_t - \theta_{xx} + T_0(\alpha\psi_t)_x &= 0 \quad \text{in } (0,L)\times(0,\infty). && (2.1.4)
\end{aligned}$$

With one of the following boundary conditions

$$\begin{aligned}
\varphi(x,t) = \psi_x(x,t) = \omega_x(x,t) = \theta(x,t) &= 0 \quad \text{for } x = 0, L, && (2.1.5)\\
\varphi(x,t) = \psi(x,t) = \omega(x,t) = \theta(x,t) &= 0 \quad \text{for } x = 0, L, && (2.1.6)
\end{aligned}$$

and initial conditions

$$\begin{cases} \varphi(x,0)=\varphi_0(x),\ \varphi_t(x,0)=\varphi_1(x),\ \ \psi(x,0)=\psi_0(x),\ \ \psi_t(x,0)=\psi_1(x) \\ \omega(x,0)=\omega_0(x),\ \ \omega_t(x,0)=\omega_1(x),\ \ \theta(x,0)=\theta_0(x) \end{cases} \tag{2.1.7}$$

where φ, ψ, ω are the vertical, shear angle and longitudinal displacements; θ is the temperature deviation from the reference temperature T_0 along the shear angle displacement and $\alpha \in W^{2,\infty}(0;L)$ is a function verifying the following condition

$$\alpha \geq 0 \text{ on }]0;L[\text{ and } \alpha \geq \alpha_0 > 0 \text{ on }]a_0;b_0[\subset]0;L[. \tag{2.1.8}$$

Here $\rho_1 = \rho A$, $\rho_2 = \rho I$, $\rho_3 = \rho c$, $\kappa_0 = EA$, $\kappa = \kappa' GA$, $b = EI$ and $l = R^{-1}$ are positive constants for the elastic and thermal material properties. To be more precise, ρ for density, E for the modulus of elasticity, G for the shear modulus, κ' for the shear factor, A for the cross-sectional area, I for the second moment of area of cross-section, R for the radius of the curvature and c for the thermal material property (for more details see Lagnese et al. [16]). The velocities of waves propagation are, respectively, $v_1 = \frac{\kappa}{\rho_1}$, $v_2 = \frac{b}{\rho_2}$, $v_3 = \frac{\kappa_0}{\rho_1}$.

The energy of solutions of the system (2.1.1)-(2.1.4) subject to initial state (2.1.7) to either the boundary conditions (2.1.5) or (2.1.6) is defined by

$$\begin{aligned} E(t) = \frac{1}{2}\int_0^L \{\kappa|\psi+\varphi_x+l\omega|^2 + b|\psi_x|^2 + \kappa_0|\omega_x - l\varphi|^2 + \rho_1|\varphi_t|^2 \\ +\rho_2|\psi_t|^2 + \rho_1|\omega_t|^2 + \frac{\rho_3}{T_0}|\theta|^2\}dx. \end{aligned} \tag{2.1.9}$$

then a straightforward computation gives :

$$\frac{d}{dt}E(t) = -\frac{1}{T_0}\int_0^L |\theta_x|^2 dx \leq 0. \tag{2.1.10}$$

Then the thermoelastic Bresse system is dissipative in the sense that its energy is non increasing with respect to the time t. Our goal is to study the effect of this dissipation on the Bresse system.

Different types of damping have been introduced to Bresse system and several uniform and polynomial stability results have been obtained. We start by recall some results related to the stabilization of elastic Bresse system. Wehbe and Youssef [34], considered elastic Bresse system subject to two locally internal dissipation laws. They proved that the system is exponentially stable if and only if the wave propagation speeds are equal. Otherwise, only a polynomial stability holds. Alabau-Boussouira et al. [2], considered the same system with one globally distributed dissipation law. The authors proved that, in general, the system is not exponentially stable but there exists polynomial decay with rates that depend on some particular relation between the coefficients. Using boundary conditions of Dirichlet-Dirichlet-Dirichlet type, they proved that the energy of the system decays at a rate $t^{-\frac{1}{3}}$ and at the rate $t^{-\frac{2}{3}}$ if $\kappa = \kappa_0$. These results are completed by Fatori and Montiero [12]. Using boundary conditions of Dirichlet-Neumann-Neumann type, the authors showed that the energy of the elastic Bresse system decays polynomially at the rate $t^{-\frac{1}{2}}$ and at the rate t^{-1}if $\kappa = \kappa_0$. Noun and Wehbe [26] extended the results of [2] and [12]. The authors considered the elastic Bresse system subject to one locally distributed feedback with Dirichlet-Neumann-Neumann or Dirichlet-Dirichlet-Dirichlet boundary conditions type. They proved that the exponentially decay rate is preserved when the wave propagation speeds are equal. On the contrary, the authors established a polynomial energy decay with rates that depend on some particular relation between the coefficients and they obtained the rate of $t^{-\frac{1}{2}}$ or t^{-1}. Finally, see [32] for the stabilization of elastic Bresse system with internal indefinite damping and [17] for the stabilization of elastic Bresse system with a nonlinear damping acting in the equation of the shear angle displacement, and nonlinear localized damping in other equations.

For the thermoelastic Bresse system, subject of this chapter, there exist two important results. The first result is due to Liu and Rao [22], when they considered the Bresse system with two thermal dissipation laws. The authors showed that the energy decays exponentially when the wave speed of the vertical displacement coincides with the wave speed of longitudinal displacement or of the shear angle displacement. Otherwise, they found polynomial decay rates depending on the boundary conditions. When the system is subject to Dirichlet-Neumann-Neumann boundary conditions, they showed that the energy decays at the rate $t^{-\frac{1}{2}}$ and for fully Dirichlet boundary conditions, they proved that the energy of the system decays as $t^{-\frac{1}{4}}$. This result has been recently improved by Fatori and Rivera [13] in the sense that the authors considered only one globally dissipative mechanism given by one temperature, and they established the rate of decay $t^{-\frac{1}{3}}$ for Dirichlet-Neumann- Neumann and Dirichlet-Dirichlet-Dirichlet boundary conditions type. The main result of this chapter is to extend the results from [13], by taking into consideration the important case when the thermal dissipation law is locally distributed on the angle displacement equation i.e the damping coefficient α is not constant but it is a positive function in $W^{2,\infty}(0,L)$ and strictly positive in an open subinterval $]a_0, b_0[\subset]0, L[$ (the cases $a_0 = 0$ or $b_0 = L$ are not excluded) and to improve the polynomial energy decay rate. Then, in this chapter, we consider the Bresse system damped by one thermal dissipation law acting locally on the angle displacement equation with Dirichlet-Neumann-Neumann or Dirichlet-Dirichlet-Dirichlet boundary conditions types. Under the equal speed wave propagation condition, $\kappa = \kappa_0$ and $\frac{\rho_1}{\rho_2} = \frac{\kappa}{b}$, using a frequency domain approach combining with a piecewise multiplier method, we establish an exponential energy decay rate for usual initial data. On the contrary, in the natural case, when $\kappa \neq \kappa_0$ and $\frac{\rho_1}{\rho_2} \neq \frac{\kappa}{b}$, we establish a new polynomial energy decay rate of type $t^{-\frac{1}{2}}$ for smooth solution. Finally, if $\kappa = \kappa_0$ and $\frac{\rho_1}{\rho_2} \neq \frac{\kappa}{b}$, we establish a new polynomial energy decay rate of type t^{-1} for the smooth solution.

We now outline briefly the content of this paper. In section 2, in a convenable

Hilbert space, we formulate system (2.1.1)-(2.1.4) with either boundary condition (2.1.5) or (2.1.6) into an evolution equation. We recall the well-posedness of the problem by the semigroup approach and by a spectrum method we prove that system (2.1.1)-(2.1.4) is strongly stable for usual initial data. In section 3, we consider the particular case when the speed of the three waves are equal and we establish an exponential energy decay rate for usual initial data. In section 4, we consider the natural general case when the speed wave propagations are different two by two and we establish a new polynomial energy decay rate for smooth initial data.

2.2 Well-Posedness and strong stability

In this section we study the existence, uniqueness and the strong stability of the solution of (2.1.1)-(2.1.7).

2.2.1 The semigroup setting.

We start by study the existence and uniqueness of the solution of the thermoelastic Bresse system. Following the two types of boundary conditions, we define the following energy spaces

$$\mathcal{H}_1 = H_0^1 \times (H_*^1)^2 \times (L^2)^2 \times L_*^2 \times L^2 \text{ and } \mathcal{H}_2 = (H_0^1)^3 \times (L^2)^4,$$

where

$$L_*^2 = \{f \in L^2(0,L) : \int_0^L f(x)dx = 0\} \text{ and } H_*^1 = \{f \in H^1(0,L) : \int_0^L f(x)dx = 0\}.$$

Both spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are equipped with the inner product which induces the energy norm

$$\|U\|^2_{\mathcal{H}_j} = \kappa\|\varphi_x + \psi + l\omega\|^2 + b\|\psi_x\|^2 + \kappa_0\|\omega_x - l\varphi\|^2$$
$$+\rho_1\|u\|^2 + \rho_2\|v\|^2 + \rho_1\|z\|^2 + \frac{\rho_3}{T_0}\|\theta\|^2. \tag{2.2.1}$$

Here and after, $\|.\|$ denotes the $L^2(0, L)$ norm.

Remark 2.2.1. *In the case of boundary condition (2.1.6), it is easy to see that expression (2.2.1) define a norm on the energy space $\mathcal{H}_2$. But in the case of boundary condition (2.1.5) the expression (2.2.1) define a norm on the energy space $\mathcal{H}_1$ if $L \neq \frac{n\pi}{l}$ for all positive integer n. Then, here and after, we assume that there exist no $n \in \mathbb{N}$ such that $L = \frac{n\pi}{l}$ when $j = 1$.*

Next, define a linear unbounded operator $\mathcal{A}_j : D(\mathcal{A}_j) \to \mathcal{H}_j$ by

$$D(\mathcal{A}_1) = \{U \in \mathcal{H}_1 : \varphi, \theta \in H_0^1 \cap H^2,\ \ \psi, \omega \in H_*^1 \cap H^2,\ \ u, \psi_x, \omega_x \in H_0^1,\ \ v, z \in H_*^1\} \tag{2.2.2}$$

$$D(\mathcal{A}_2) = \{U \in \mathcal{H}_2 : \varphi, \psi, \omega, \theta \in H_0^1 \cap H^2,\ \ u, v, z \in H_0^1\} \tag{2.2.3}$$

$$\mathcal{A}_j(\varphi, \psi, \omega, u, v, z, \theta) = \begin{pmatrix} u \\ v \\ z \\ \frac{\kappa}{\rho_1}(\varphi_x + \psi + l\omega)_x + \frac{\kappa_0 l}{\rho_1}(\omega_x - l\varphi) \\ \frac{b}{\rho_2}\psi_{xx} - \frac{\kappa}{\rho_2}(\varphi_x + \psi + l\omega) - \frac{1}{\rho_2}\alpha(x)\theta_x \\ \frac{\kappa_0}{\rho_1}(\omega_x - l\varphi)_x - \frac{\kappa l}{\rho_1}(\varphi_x + \psi + l\omega) \\ \frac{1}{\rho_3}\theta_{xx} - \frac{T_0}{\rho_3}(\alpha v)_x \end{pmatrix} \tag{2.2.4}$$

for all $U = (\varphi, \psi, \omega, u, v, z, \theta) \in D(\mathcal{A}_j)$, $j = 1, 2$.

Thus, if $U = (\varphi, \psi, \omega, \varphi_t, \psi_t, \omega_t, \theta)$ is a smooth solution of system (2.1.1)-(2.1.7), then the thermoelastic Bresse system is transformed into a first order evolution equation on the Hilbert space $\mathcal{H}_j$:

$$U_t = \mathcal{A}_j U, \quad U(0) = U_0 \tag{2.2.5}$$

with $j = 1, 2$ corresponding to the boundary conditions (2.1.6) and (2.1.7), respectively.

It is easy to see that the operator $\mathcal{A}_j$ is m-dissipative in the energy space $\mathcal{H}_j$, $j = 1, 2$, then we have the following results concerning existence and uniqueness of solution of the problem (2.2.5) (see [27], [23]).

Theorem 2.2.2. *The operator $\mathcal{A}_j$ generates a C_0-semigroup e^{tA_j} of contractions on $\mathcal{H}_j$ for $j = 1, 2$. Thus for any initial data $U_0 \in \mathcal{H}_j$, the problem (2.2.5) has a unique weak solution $U \in \mathcal{C}^0([0, \infty), \mathcal{H}_j)$. Moreover, if $U_0 \in D(\mathcal{A}_j)$, then U is a strong solution of (2.2.5), i. e $U \in \mathcal{C}^1([0, \infty), \mathcal{H}_j) \cap \mathcal{C}^0([0, \infty), D(\mathcal{A}_j))$.*

2.2.2 Strong stability result.

In this part, using a spectrum method, we will prove the strong stability of the C_0-semigroup $e^{t\mathcal{A}_j}$.

Theorem 2.2.3. *The semigroup e^{tA_j} is strongly stable in the energy space $\mathcal{H}_j$. In other words*

$$\lim_{t\to+\infty} \|e^{tA_j} U_0\|_{\mathcal{H}_j} = 0 \quad j = 1,\ 2 \tag{2.2.6}$$

for all $U_0 \in \mathcal{H}_j$.

Proof. Since the resolvent of $\mathcal{A}_j$ is compact in $\mathcal{H}_j$, $j = 1, 2$, then using a result due to Benchimol [7], the system (2.1.1)-(2.1.4) is strongly stable if and only if $\mathcal{A}_j$ does not have pure imaginary eigenvalues. By contradiction argument, let $0 \neq U = (\varphi, \psi, \omega, u, v, z, \theta) \in D(\mathcal{A}_j)$, $i\lambda \in i\mathbb{R}$, such that

$$\mathcal{A}_j U = i\lambda U.$$

Our goal is to find a contradiction by proving that $U = 0$. Taking the real part of the inner product in $\mathcal{H}_j$ of $\mathcal{A}_j U$ and U, we get

$$0 = Re(i\lambda ||U||^2_{\mathcal{H}_j}) = Re((\mathcal{A}_j U, U)_{\mathcal{H}_j}) = -\frac{1}{T_0}\int_0^L |\theta_x|^2 dx.$$

It follows that

$$\theta = \theta_x = 0 \qquad \text{a.e} \quad \text{in} \quad (0, L).$$

Now, detailing the equation $\mathcal{A}_j U = i\lambda U$, and using the fact that $\theta = 0$, we get

$$u = i\lambda\varphi, \tag{2.2.7}$$

$$v = i\lambda\psi, \tag{2.2.8}$$

$$z = i\lambda\omega, \tag{2.2.9}$$

$$\frac{\kappa}{\rho_1}(\varphi_x + \psi + l\omega)_x + \frac{\kappa_0 l}{\rho_1}(\omega_x - l\varphi) = i\lambda u, \tag{2.2.10}$$

$$\frac{b}{\rho_2}\psi_{xx} - \frac{\kappa}{\rho_2}(\varphi_x + \psi + l\omega) = i\lambda v, \tag{2.2.11}$$

$$\frac{\kappa_0}{\rho_1}(\omega_x - l\varphi)_x - \frac{\kappa l}{\rho_1}(\varphi_x + \psi + l\omega) = i\lambda z, \tag{2.2.12}$$

$$(\alpha v)_x = 0. \tag{2.2.13}$$

If $\lambda = 0$, then $u = v = z = 0$ and using Lax-Milgram theorem (see [10]), it is clear to see that the system (2.2.10)-(2.2.12) has the unique trivial solution $\varphi = \psi = \omega = 0$. This implies that $U = 0$ and the desired contradiction is proved.
Now, assume that $\lambda \neq 0$. Then let $\xi(x) = \int_0^x v(s)ds$, multiply equation (2.2.13) by $-\overline{\xi(x)}$, and integrate by parts, we get

$$\int_0^L \alpha |v|^2 dx - \alpha(L)v(L)\int_0^L v(s)ds = 0.$$

In the case of Dirichlet-Neumann-Neumann conditions, we have $v \in H^1_*(0, L)$ then $\int_0^L v(s)ds = 0$, and in the case of Dirichlet- Dirichlet -Dirichlet conditions, we have $v \in H^1_0(0, L)$ then $v(L) = 0$. This together with condition (2.1.8), implies that

$$\sqrt{\alpha} v = 0 \text{ a.e in } (0, L) \qquad \text{and} \qquad v = 0 \text{ a.e in } (a_0, b_0). \tag{2.2.14}$$

Now, combining equations (2.2.8), (2.2.11) and (2.2.14), we get

$$\psi = 0 \text{ and } \varphi_x + l\omega = 0 \qquad \text{a.e in } (a_0, b_0). \tag{2.2.15}$$

Combining equations (2.2.7), (2.2.10) and (2.2.15), we get

$$\rho_1 \lambda^2 \varphi + \kappa_0 l(\omega_x - l\varphi) = 0, \qquad \text{a.e in } (a_0, b_0). \tag{2.2.16}$$

Similarly, combining equations (2.2.9), (2.2.12) and (2.2.15), we get

$$\rho_1 \lambda^2 \omega + \kappa_0 (\omega_x - l\varphi)_x = 0, \qquad \text{a.e in } (a_0, b_0). \tag{2.2.17}$$

By a direct calculation we deduce that system (2.2.15)-(2.2.17) has the following solution

$$\varphi = c, \ \ \psi = 0, \ \ \omega = 0, \qquad \text{a.e in } (a_0, b_0).$$

Then, from equation (2.2.16) we deduce that

$$(\lambda^2 \rho_1 - \kappa_0 l^2)\varphi = 0. \qquad \text{a.e in } (a_0, b_0).$$

We are against two cases to discuss, $\lambda = l\sqrt{\frac{\kappa_0}{\rho_1}}$, or $\lambda \neq l\sqrt{\frac{\kappa_0}{\rho_1}}$.

Case 1. Suppose that $\lambda \neq l\sqrt{\frac{\kappa_0}{\rho_1}}$, then

$$\varphi = 0 \qquad \text{a.e} \quad \text{in } (a_0, b_0).$$

Let $X = (\varphi, \varphi_x, \psi, \psi_x, \omega, \omega_x)^T$ and

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{-\rho_1}{\kappa}\lambda^2 + \frac{\kappa_0}{\kappa} l^2 & 0 & 0 & -1 & 0 & -l - \frac{\kappa_0}{\kappa} l \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{\kappa}{b} & \frac{-\rho_2}{b}\lambda^2 + \frac{\kappa}{b} & 0 & \frac{\kappa}{b} l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & l + \frac{\kappa}{\kappa_0} l & \frac{\kappa}{\kappa_0} l & 0 & \frac{-\rho_1}{\kappa_0}\lambda^2 + \frac{\kappa}{\kappa_0} l^2 & 0 \end{pmatrix}.$$

Then system (2.2.10)-(2.2.12) could be given as

$$\begin{cases} X' = MX, \text{ in } (0, a_0), \\ X(a_0) = 0. \end{cases} \tag{2.2.18}$$

Using ordinary differential equation theory, we deduce that system (2.2.18) has the unique trivial solution $X = 0$ in $(0, a_0)$ and $\varphi = \psi = \omega = 0$ a.e in $(0, a_0)$. Same argument as above leads us to prove that $\varphi = \psi = \omega = 0$ a.e in (b_0, L) and therefore $U = 0$.

Case 2. Suppose that $\lambda = l\sqrt{\frac{\kappa_0}{\rho_1}}$. Then equation (2.2.10) can be rewritten as

$$\kappa(\varphi_x + \psi + l\omega)_x + \frac{\kappa_0 l}{\kappa}\omega_x = 0 \qquad \text{a.e} \quad \text{in } (0, a_0). \tag{2.2.19}$$

Let $X = (\varphi_x, \psi, \psi_x, \omega, \omega_x)^T$ and

$$M = \begin{pmatrix} 0 & 0 & -1 & 0 & -l - \frac{\kappa_0}{\kappa^2}l \\ 0 & 0 & 1 & 0 & 0 \\ \frac{\kappa}{b} & \frac{-\rho_2}{b}\lambda^2 + \frac{\kappa}{b} & 0 & \frac{\kappa}{b}l & 0 \\ 0 & 0 & 0 & 0 & 1 \\ l + \frac{\kappa}{\kappa_0}l & \frac{\kappa}{\kappa_0}l & 0 & \frac{-\rho_1}{\kappa_0}\lambda^2 + \frac{\kappa}{\kappa_0}l^2 & 0 \end{pmatrix}$$

Then system (2.2.10)-(2.2.12) could be given as

$$\begin{cases} X' = MX, \text{ in } (0, a_0), \\ X(a_0) = 0. \end{cases} \tag{2.2.20}$$

Using ordinary differential equation theory, we deduce that system (2.2.20) has the unique trivial solution $X = 0$ in $(0, a_0)$. This implies that $\varphi = c$, $\psi = 0$ and $\omega = 0$ a.e in $(0, a_0)$. Since $\varphi \in H^2(0, L) \subset C^1([0, L])$ and $\varphi(0) = 0$, we conclude that $\varphi = 0$ a.e in $(0, a_0)$. Same argument as above leads us to prove that $\varphi = \psi = \omega = 0$ a.e in (b_0, L) and therefore $U = 0$. The proof is thus complete. $\square$

2.3 Exponential Stability, the case of $\kappa = \kappa_0$ and $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$.

In this section, we consider system (2.1.1)-(2.1.4) under the equal speed propagation conditions i.e. $\kappa = \kappa_0$ and $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$. We prove the following exponential stability result:

Theorem 2.3.1. *If $\kappa = \kappa_0$ and $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$ then the semigroup $e^{t\mathcal{A}_j}$ is exponentially stable, i.e., there exist constant $M \geq 1$, and $\epsilon > 0$ independent of U_0 such that*

$$\|e^{t\mathcal{A}_j}U_0\|_{\mathcal{H}_j} \leq Me^{-\epsilon t}\|U_0\|_{\mathcal{H}_j},\ t \geq 0 \quad j = 1,\, 2. \tag{2.3.1}$$

For this aim, we will use the frequency domain method. More precisely, using Huang [15] and Prüss [28], inequality (2.3.1) hold if and only if the following conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}_j) \tag{H1}$$

and

$$\sup_{\lambda\in\mathbb{R}}\|(i\lambda I - \mathcal{A}_j)^{-1}\| = O(1) \tag{H2}$$

are true. We first check condition (H1). Since $(I - \mathcal{A}_j)^{-1}$ is compact and $\mathcal{A}_j$ has no pure imaginary eigenvalues (Theorem 2.3), we deduce that condition (H1) is true. We will prove condition (H2) by contradiction argument. Suppose that there exist a sequence $\lambda_n \in \mathbb{R}$ and a sequence $U^n = (\varphi^n, \psi^n, \omega^n, u^n, v^n, z^n, \theta^n) \in D(\mathcal{A}_j)$, verifying the following conditions

$$|\lambda_n| \longrightarrow +\infty, \tag{2.3.2}$$

$$\|U^n\|_{\mathcal{H}_j} = 1, \tag{2.3.3}$$

$$(i\lambda_n I - \mathcal{A}_j)U^n = (f_1^n, f_2^n, f_3^n, g_1^n, g_2^n, g_3^n, g_4^n) \longrightarrow 0 \quad \text{in} \quad \mathcal{H}_j, \quad j = 1,\, 2. \tag{2.3.4}$$

Equation (2.3.4) could be written as

$$i\lambda_n\varphi^n - u^n = f_1^n \tag{2.3.5}$$

$$i\lambda_n\psi^n - v^n = f_2^n \tag{2.3.6}$$

$$i\lambda_n\omega^n - z^n = f_3^n \tag{2.3.7}$$

$$\lambda_n^2\varphi^n + \frac{\kappa}{\rho_1}(\varphi_{xx}^n + \psi_x^n + l\omega_x^n) + \frac{\kappa_0 l}{\rho_1}(\omega_x^n - l\varphi^n) = -g_1^n - i\lambda_n f_1^n, \tag{2.3.8}$$

$$\lambda_n^2\psi^n + \frac{b}{\rho_2}\psi_{xx}^n - \frac{\kappa}{\rho_2}(\varphi_x^n + \psi^n + l\omega^n) - \frac{1}{\rho_2}\alpha(x)\theta_x^n = -g_2^n - i\lambda_n f_2^n, \tag{2.3.9}$$

$$\lambda_n^2\omega^n + \frac{\kappa_0}{\rho_1}(\omega_{xx}^n - l\varphi_x^n) - \frac{\kappa l}{\rho_1}(\varphi_x^n + \psi^n + l\omega^n) = -g_3^n - i\lambda_n f_3^n \tag{2.3.10}$$

$$i\lambda_n\theta^n - \frac{1}{\rho_3}\theta_{xx}^n + i\frac{T_0}{\rho_3}\lambda_n(\alpha\psi^n)_x = g_4^n + \frac{T_0(\alpha f_2^n)_x}{\rho_3}. \tag{2.3.11}$$

Our goal is, using a multiplier method, to prove that $\|U\|_{\mathcal{H}_j} = o(1)$. This contradicts equation (2.3.3). We will establish the proof by several Lemmas. For simplicity, here and after we drop the index n.

Consider the function $\eta \in C^1([0, L])$ such that $0 \leq \eta \leq 1$, $\eta = 1$ on $[a_0 + \varepsilon, b_0 - \varepsilon]$ and $\eta = 0$ on $[0, a_0] \cup [b_0, L]$, where $0 < a_0 + \varepsilon < b_0 - \varepsilon < L$.

Lemma 2.3.2. *(First information)*

Under the above notations we have

$$\|\psi_x\| = O(1), \quad \|\psi\| = \frac{O(1)}{\lambda} \quad \text{and} \quad \|\eta\psi_{xx}\| = O(\lambda). \tag{2.3.12}$$

Proof. Using equations (2.3.5), (2.3.6), (2.3.7) and (2.3.9), we deduce equations (2.3.12). □

Lemma 2.3.3. *(The dissipation)*

Under the above notations we have

$$\int_0^L |\theta_x|^2 dx = o(1) \quad \text{and} \quad \int_0^L |\theta|^2 dx = o(1). \tag{2.3.13}$$

Proof. Multiplying equation (2.3.7) by the uniformly bounded sequence $U = (\varphi, \psi, \omega, u, v, z, \theta)$, we get

$$\int_0^L |\theta_x|^2 dx = -\mathrm{Re}((i\lambda - \mathcal{A}_j)U, U)_{\mathcal{H}_j} = o(1). \tag{2.3.14}$$

Finally, using Poincaré inequality, it follows the second asymptotic equality. The proof is thus completed. □

For the next lemma, define the spaces

$$\mathcal{H}_{1a,b} := H_0^1(a,b) \times (H_*^1(a,b))^2 \times (L^2(a,b))^2 \times L_*^2(a,b) \times L^2(a,b),$$

and

$$\mathcal{H}_{2a,b} := (H_*^1(a,b))^3 \times (L^2(a,b))^4$$

Lemma 2.3.4. *Under the above notations, if* $||U||_{\mathcal{H}_{ja,b}} = o(1), j = 1, 2$ *for some* $0 < a < b < L$, *then* $||U||_{\mathcal{H}_j} = o(1)$.

Proof. Let $h \in H_0^1(0; L)$ be a given function.
(i) Multiply equation (2.3.8) by $2\rho_1 h\overline{\varphi_x}$ and integrate over $[0; L]$, we get

$$\begin{aligned} -\rho_1 \int_0^L h'|\lambda\varphi|^2 + \rho_1[h|\lambda\varphi|^2]_0^L - \kappa \int_0^L h'|\varphi_x|^2 + \kappa[h|\varphi_x|^2]_0^L \\ +2Re\left\{\kappa \int_0^L h\psi_x\overline{\varphi_x} + l(\kappa+\kappa_0)\int_0^L h\omega_x\overline{\varphi_x} - \kappa_0 l^2 \int_0^L h\varphi\overline{\varphi_x}\right\} \\ = 2\rho_1 Re\left\{\int_0^h g_1\overline{\varphi_x} + i\int_0^L (f_{1x}h + f_1 h')\lambda\overline{\varphi} - i\lambda[f_1 h\overline{\varphi}]_0^L\right\}. \end{aligned} \tag{2.3.15}$$

Using equations (2.3.3) and (2.3.5), we deduce that $||\varphi|| = \frac{O(1)}{\lambda}$ and $||\varphi_x|| = O(1)$. Then using the fact that $\varphi(0) = \varphi(L) = 0$, $h(0) = h(L) = 0$, $||g_1|| = o(1)$, $||f_1|| = o(1)$ and $||f_{1x}|| = o(1)$ in (2.3.15), we get

$$\begin{aligned} -\rho_1 \int_0^L h'|\lambda\varphi|^2 - \kappa \int_0^L h'|\varphi_x|^2 \\ +2Re\left\{\kappa \int_0^L h\psi_x\overline{\varphi_x} + l(\kappa+\kappa_0)\int_0^L h\omega_x\overline{\varphi_x}\right\} = o(1). \end{aligned} \tag{2.3.16}$$

(ii) Multiply equation (2.3.9) by $2\rho_2 h\overline{\psi_x}$ and integrate over $[0;L]$, we get

$$\begin{aligned} -\rho_2\int_0^L h'|\lambda\psi|^2 + \rho_2[h|\lambda\psi|^2]_0^L - b\int_0^L h'|\psi_x|^2 + b[h|\psi_x|^2]_0^L \\ -2Re\left\{\kappa\int_0^L h\varphi_x\overline{\psi_x} + \kappa\int_0^L h\psi\overline{\psi_x} + \kappa l\int_0^L h\omega\overline{\psi_x} + \int_0^L h\alpha(x)\theta_x\overline{\psi_x}\right\} \\ = 2\rho_2 Re\left\{-\int_0^L hg_2\overline{\psi_x} + i\int_0^L (f_{2x}h + f_2h')\lambda\overline{\psi} - i\lambda[f_2h\overline{\psi}]_0^L\right\}. \end{aligned} \tag{2.3.17}$$

Using equations (2.3.3), (2.3.6) and (2.3.7) we deduce that $||\psi|| = \frac{O(1)}{\lambda}$, $||\omega|| = \frac{O(1)}{\lambda}$ and $||\psi_x|| = O(1)$. Then using the fact that $h(0) = h(L) = 0$, $||\theta_x|| = o(1)$, $||g_2|| = o(1)$, $||f_2|| = o(1)$ and $||f_{2x}|| = o(1)$ in equation (2.3.17), we get

$$-\rho_2\int_0^L h'|\lambda\psi|^2 - b\int_0^L h'|\psi_x|^2 - 2\kappa Re\left\{\int_0^L h\varphi_x\overline{\psi_x}\right\} = o(1). \tag{2.3.18}$$

(iii) Similarly, multiply equation (2.3.10) by $2\rho_1 h\overline{\omega_x}$ and integrate over $[0;L]$, we get

$$\begin{aligned} -\rho_1\int_0^L h'|\lambda\omega|^2 + \rho_1[h|\lambda\omega|^2]_0^L - \kappa_0\int_0^L h'|\omega_x|^2 + \kappa_0[h|\omega_x|^2]_0^L \\ -2lRe\left\{\kappa_0\int_0^L h\varphi_x\overline{\omega_x} + \kappa\int_0^L h\varphi_x\overline{\omega_x} + \kappa\int_0^L h(\psi + l\omega)\overline{\omega_x}\right\} \\ = 2\rho_1 Re\left\{-\int_0^L hg_3\overline{\omega_x} + i\int_0^L (f_{3x}h + f_3h')\lambda\overline{\omega} - i\lambda[f_3h\overline{\omega}]_0^L\right\}. \end{aligned} \tag{2.3.19}$$

From a similar way as in (i) and (ii), it follows that

$$-\rho_1\int_0^L h'|\lambda\omega|^2 - \kappa_0\int_0^L h'|\omega_x|^2 - 2l(\kappa+\kappa_0)Re\left\{\int_0^L h\varphi_x\overline{\omega_x}\right\} = o(1). \tag{2.3.20}$$

(iv) Adding equations (2.3.16), (2.3.18) and (2.3.20), we get

$$\begin{aligned} -\rho_1\int_0^L h'|\lambda\varphi|^2 - \kappa\int_0^L h'|\varphi_x|^2 - \rho_2\int_0^L h'|\lambda\psi|^2 - b\int_0^L h'|\psi_x|^2 \\ -\rho_1\int_0^L h'|\lambda\omega|^2 - \kappa_0\int_0^L h'|\omega_x|^2 = o(1). \end{aligned} \tag{2.3.21}$$

(v) Let $\varepsilon > 0$ such that $a + \varepsilon < b$ and define the function $\widehat{\eta}$ in $C^1([0;L])$ by:

$$0 \le \widehat{\eta} \le 1,\ \ \widehat{\eta} = 1 \ \text{ on } \ [0;a] \ \text{ and } \ \widehat{\eta} = 0 \ \text{ on } \ [a+\varepsilon;L]$$

Then take $h = x\widehat{\eta}$ in (2.3.21) and using the fact that $\|U\|_{\mathcal{H}_{ja,b}} = o(1)$, we get

$$\begin{aligned}-\rho_1\int_0^a|\lambda\varphi|^2-\kappa\int_0^a|\varphi_x|^2-\rho_2\int_0^a|\lambda\psi|^2-b\int_0^a|\psi_x|^2\\ -\rho_1\int_0^a|\lambda\omega|^2-\kappa_0\int_0^a|\omega_x|^2=o(1).\end{aligned}\tag{2.3.22}$$

It follows that

$$\|U\|_{\mathcal{H}_{j0,a}} = o(1).$$

(vi) Let $\varepsilon > 0$ such that $b - \varepsilon > a$ and define the function $\widetilde{\eta}$ in $C^1([0; L])$ by:

$$0 \le \widetilde{\eta} \le 1, \;\; \widetilde{\eta} = 1 \;\text{ on }\; [b, L] \;\text{ and }\; \widetilde{\eta} = 0 \;\text{ on }\; [0, b - \varepsilon].$$

Then, by a similar way used in (v), take $h = (x - L)\widetilde{\eta}$ in (2.3.21) and using the fact that $\|U\|_{\mathcal{H}_{ja,b}} = o(1)$, we get

$$\|U\|_{\mathcal{H}_{jb,L}} = o(1).$$

The proof is thus completed. □

Lemma 2.3.5. *(Information on ψ and ψ_x)*
Under the above notations, we have

$$\int_0^L \eta|\psi|^2 = \frac{o(1)}{\lambda^2} \quad \text{and} \quad \int_0^L \eta|\psi_x|^2 = o(1). \tag{2.3.23}$$

Proof. First, multiplying equation (2.3.11) by $\eta\bar{\psi}_x$, we get

$$\begin{aligned}T_0\int_0^L\eta\alpha|\psi_x|^2=\frac{T_0}{2}\int_0^L(\eta\alpha')'|\psi|^2\\ +Re\left\{\rho_3\int_0^L(\eta'\theta+\eta\theta_x)\bar{\psi}+i\int_0^L\theta_x\lambda^{-1}\eta\bar{\psi}_{xx}+\frac{i}{\lambda}\int_0^L\eta'\theta_x\bar{\psi}_x\right\}+\frac{o(1)}{\lambda}.\end{aligned}\tag{2.3.24}$$

Using equation (2.3.13) and the fact that $\|\psi\| = \frac{O(1)}{\lambda}$, $\|\psi_x\| = O(1)$ and $\|\eta\psi_{xx}\| = O(\lambda)$ in (2.3.24), we get

$$\int_0^L \eta|\psi_x|^2 = o(1). \tag{2.3.25}$$

Next, multiplying equation (2.3.9) by $\eta\bar{\psi}$, we get

$$\rho_2 \int_0^L \eta|\lambda\psi|^2 = b\int_0^L \eta|\psi_x|^2 + b\int_0^L \eta'\psi_x\bar{\psi} + \int_0^L [\kappa(\psi + l\omega) + \alpha\theta_x]\eta\bar{\psi} \\ - \int_0^1 \kappa(\eta'\varphi\psi + \eta\varphi\psi_x) + o(1). \tag{2.3.26}$$

Using equation (2.3.13), (2.3.25) and the fact that $\|\psi\| = \frac{O(1)}{\lambda}$ and $\|\omega\| = \frac{O(1)}{\lambda}$ in equation (2.3.26), we get

$$\int_0^L \eta|\psi|^2 = \frac{o(1)}{\lambda^2}. \tag{2.3.27}$$

□

Lemma 2.3.6. *(Information on φ and φ_x)*
Under the above notations, if $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$, then we have

$$\int_0^L \eta|\varphi|^2 = \frac{o(1)}{\lambda^2} \quad \text{and} \quad \int_0^L \eta|\varphi_x|^2 = o(1). \tag{2.3.28}$$

Proof. (i) First, multiplying equation (2.3.8) by $\eta\overline{\psi_x^n}$ and integrating over $]0, L[$, we get

$$\int_0^L \eta\lambda^2\varphi\overline{\psi_x} + \frac{\kappa}{\rho_1}\int_0^L \eta\varphi_{xx}\overline{\psi_x} + \frac{\kappa}{\rho_1}\int_0^L \eta|\psi_x|^2 \\ + \frac{\kappa l}{\rho_1}\int_0^L \eta\omega_x\overline{\psi_x} + \frac{\kappa_0 l}{\rho_1}\int_0^L (\omega_x - l\varphi)\eta\overline{\psi_x} \\ = \int_0^L (-g_1\eta\overline{\psi_x} + i\lambda f_{1x}\eta\overline{\psi} + i\lambda f_1\eta'\overline{\psi}) - \left[i\lambda f_1\eta\overline{\psi}\right]_0^L. \tag{2.3.29}$$

From equations (2.3.3), (2.3.5) and (2.3.6) it is clear to see that sequences ω_x, $(\omega_x - l\varphi)$, $\lambda\psi$ are uniformly bounded in $L^2(0, L)$. Then using Lemma 2.3.5 and the fact that $\|f_1\| = o(1)$, $\|f_{1x}\| = o(1)$, $\|g_1\| = o(1)$, and that $f_1(0) = f_1(L) = 0$, we obtain the following equation

$$-\int_0^L \eta\lambda^2\varphi\overline{\psi_x} - \frac{\kappa}{\rho_1}\int_0^L \eta\varphi_{xx}\overline{\psi_x} = o(1). \tag{2.3.30}$$

(ii) Next, multiply equation (2.3.9) by $\eta\overline{\varphi_x}$ and integrate over $]0,L[$, we get

$$\begin{aligned}
-\int_0^L \lambda^2\psi_x\eta\overline{\varphi} - \int_0^L \lambda^2\psi\eta'\overline{\varphi} + \left[\lambda^2\psi\eta\overline{\varphi}\right]_0^L - \frac{b}{\rho_2}\int_0^L \psi_x\eta\overline{\varphi_{xx}} \\
-\frac{b}{\rho_2}\int_0^L \psi_x\eta'\overline{\varphi_x} + \frac{b}{\rho_2}\left[\psi_x\eta\varphi_x\right]_0^L - \frac{\kappa}{\rho_2}\int_0^L \eta|\varphi_x|^2 \\
-\frac{\kappa}{\rho_2}\int_0^L (\psi + l\omega)\eta\overline{\varphi_x} - \frac{1}{\rho_2}\int_0^L \eta\alpha(x)\theta_x\overline{\varphi_x} \\
= \int_0^L (-g_2\eta\overline{\varphi_x} + i\lambda f_{2x}\eta\overline{\varphi} + i\lambda f_2\eta'\overline{\varphi}) - \left[i\lambda f_2\eta\overline{\varphi}\right]_0^L .
\end{aligned} \tag{2.3.31}$$

Using Lemma 2.3.5 and the fact that the sequences $\lambda\varphi$, φ_x, $\alpha(x)\varphi_x$ are uniformly bounded in $L^2(0,L)$, we get

$$\int_0^L \lambda^2\psi_x\eta\overline{\varphi} + \frac{b}{\rho_2}\int_0^L \psi_x\eta\overline{\varphi_{xx}} + \frac{\kappa}{\rho_2}\int_0^L \eta|\varphi_x|^2 = o(1). \tag{2.3.32}$$

(iii) Adding the real parts of equations (2.3.30) and (2.3.32) and using the condition $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$ we get

$$\int_0^L \eta|\varphi_x|^2 = o(1) \tag{2.3.33}$$

Multiplying equation (2.3.8) by $\eta\overline{\varphi}$ and integrating over $]0,L[$, we get

$$\begin{aligned}
\rho_1\int_0^L \eta|\lambda\varphi|^2 = \kappa\int_0^L \eta|\varphi_x|^2 + \kappa\int_0^L \eta'\varphi_x\overline{\varphi} - \kappa\int_0^L (\psi_x + l\omega_x)\eta\overline{\varphi} \\
-\kappa_0 l\int_0^L (\omega_x - l\varphi)\eta\overline{\varphi} + o(1).
\end{aligned} \tag{2.3.34}$$

Using equations (2.3.33), (2.3.25), the fact that $\|\varphi\| = \frac{O(1)}{\lambda}$ and the sequences φ_x, $(\psi_x - l\omega_x)$, $(\omega_x - l\varphi)$ are uniformly bounded in $L^2(0,L)$ in equation (2.3.34), we get

$$\int_0^L \eta|\varphi|^2 = \frac{o(1)}{\lambda^2}. \tag{2.3.35}$$

The proof is thus completed. □

Lemma 2.3.7. *(Information on ω and ω_x)*
Under the above notations, if $\kappa = \kappa_0$ and $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$, then we have

$$\int_0^L \eta|\omega|^2 = \frac{o(1)}{\lambda^2} \quad \text{and} \quad \int_0^L \eta|\omega_x|^2 = o(1). \tag{2.3.36}$$

Proof. (i) First, multiply equation (2.3.8) by $\rho_1\eta\overline{\omega_x}$ and integrate over $]0, L[$, we get

$$\begin{aligned} -\rho_1 \int_0^L \lambda^2\eta\varphi_x\overline{\omega} - \kappa\int_0^L \varphi_x\eta\overline{\omega_{xx}} - \kappa\int_0^L \varphi_x\eta'\overline{\omega_x} \\ +\kappa\int_0^L \psi_x\eta\overline{\omega_x} + (\kappa+\kappa_0)l\int_0^L \eta|\omega_x|^2 - \kappa_0 l^2\int_0^L \varphi\eta\overline{\omega_x} = o(1) \end{aligned} \tag{2.3.37}$$

Using Lemma 2.3.5, Lemma 2.3.6 and the fact that $\|\omega_x\| = O(1)$ in equation (2.3.37), we get

$$-\rho_1\int_0^L \lambda^2\eta\varphi_x\overline{\omega} + (\kappa+\kappa_0)l\int_0^L \eta|\omega_x|^2 - \kappa\int_0^L \varphi_x\eta\overline{\omega_{xx}} = o(1). \tag{2.3.38}$$

(ii) Next, multiplying equation (2.3.10) by $\rho_1\eta\overline{\varphi_x}$ and integrating over $]0, L[$, we get

$$\begin{aligned} \rho_1\int_0^L \lambda^2\eta\omega\overline{\varphi_x} + \kappa_0\int_0^L \eta\omega_{xx}\overline{\varphi_x} - (\kappa+\kappa_0)l\int_0^L \eta|\varphi_x|^2 \\ -\kappa l\int_0^L (\psi + l\omega)\eta\overline{\varphi_x} = o(1). \end{aligned} \tag{2.3.39}$$

Using Lemma 2.3.5, Lemma 2.3.6 and the fact that $\|\omega\| = \frac{O(1)}{\lambda}$ in equation (2.3.39), we get

$$\rho_1\int_0^L \lambda^2\eta\omega\overline{\varphi_x} + \kappa_0\int_0^L \eta\omega_{xx}\overline{\varphi_x} = o(1). \tag{2.3.40}$$

(iii) Adding the real parts of equations (2.3.38) and (2.3.40), and using the fact that $\kappa = \kappa_0$, we deduce that

$$\int_0^L \eta|\omega_x|^2 = o(1) \tag{2.3.41}$$

Finally, by a similar way used in (iii) Lemma 2.3.6, multiplying equation (2.3.10) by $\eta\bar{\omega}$, we deduce the first asymptotic behavior equation in (2.3.36). The proof is thus completed. □

Proof of Theorem 3.1 Using Lemma 2.3.3, Lemma 2.3.5, Lemma 2.3.6 and Lemma 2.3.7, we deduce that $||U||_{\mathcal{H}_j} = o(1)$ on the subinterval $[a_0; b_0]$. Then using Lemma 2.3.4 we deduce that $||U|| = o(1)$ on the interval $[0; L]$, this contradicts equality (2.3.3). We deduce that the resolvent of the operator $\mathcal{A}_j$ is uniformly bounded on the imaginary axis $i\mathbb{R}$. This together with the fact that $i\mathbb{R} \subset \rho(\mathcal{A}_j)$ implies, under the equal speed propagation conditions, the exponential stability of system (2.1.1)-(2.1.4) with either boundary Dirichlet- Dirichlet- Dirichlet or Dirichlet-Neumann-Neumann conditions types. The proof is thus completed.

Remark 2.3.8. *From the theory of elasticity, $\rho_1 = \rho A$, $\rho_2 = \rho I$, $\kappa_0 = EA$, $\kappa = \kappa' GA$, and $b = EI$, where ρ for density, E denotes the Young's modulus of elasticity, G for the shear modulus, κ' for the shear factor, A for the cross-sectional area and I for the second moment of area of cross-section. Then the equal speed propagation conditions $\kappa = \kappa_0$ or $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$ are equivalent to $\kappa' G = E$. But the two elastic modulus are not equal since $\kappa' G = \frac{E}{2(1+\mu)}$ where $\mu \in (0, \frac{1}{2})$ is the Poisson's ratio. Thus, the exponential stability is only mathematically sound.*

2.4 Polynomial Stability, the general case

The thermoelastic Bresse system (2.1.1)-(2.1.4) with the boundary condition (2.1.5) is not exponentially stable when $\kappa \neq \kappa_0$ or $\frac{\rho_1}{\rho_2} \neq \frac{\kappa}{b}$ (see [34], [13], [2]). The idea is to find a real sequence (λ_n) with $|\lambda_n| \to \infty$ and a sequence U^n of elements of $D(\mathcal{A}_1)$ with $\|U^n\| = 1$ such that $\|(i\lambda_n - \mathcal{A}_1)U^n\| = o(1)$. Then the resolvent of the operator $\mathcal{A}_1$ is not uniformly bounded on the imaginary axes and the system is not exponentially stable (see [15], [28]). Our main results are the following polynomial-type decay rate.

Theorem 2.4.1. *(Polynomial energy decay rate) Assume that* $\kappa \neq \kappa_0$ *and* $\frac{\rho_1}{\rho_2} \neq \frac{\kappa}{b}$*. Then there exists a constant* $C > 0$ *such that for every initial data* $U_0 = (\varphi_0, \psi_0, \omega_0, \varphi_1, \psi_1, \omega_1, \theta_0) \in D(\mathcal{A}_j)$*,* $j = 1, 2$*, the energy of system (2.1.1)-(2.1.4) with boundary conditions (2.1.5) or (2.1.6) verify the following estimation:*

$$E(t) \leq C\frac{1}{\sqrt{t}}\|U_0\|^2_{D(\mathcal{A}_j)} \quad \forall t > 0. \tag{2.4.1}$$

Following Borichev and Tomilov [9], (see also [20], [6]), a C_0 semigroup of contractions $e^{t\mathcal{A}_j}$ on a Hilbert space $\mathcal{H}_j$ verify (2.4.1) if (H1) and

$$\sup_{\lambda\in\mathbb{R}} \frac{1}{|\lambda|^4}\|(i\lambda I - \mathcal{A}_j)^{-1}\| < +\infty \tag{2.4.2}$$

are satisfied. Condition (H_1) was already proved in Theorem 3.1 and 2.3. Our goal is to prove that $\|(i\lambda - \mathcal{A}_j)^{-1}\| = O(|\lambda^4|)$. By contradiction argument, suppose that there exist a sequence $\lambda_n \in \mathbb{R}$ and a sequence $U^n = (\varphi^n, \psi^n, \omega^n, u^n, v^n, z^n, \theta^n) \in D(\mathcal{A}_j)$, verifying the following conditions

$$|\lambda_n| \longrightarrow +\infty, \quad \|U^n\| = \|(\varphi^n, \psi^n, \omega^n, u^n, v^n, z^n, \theta^n)\|_{\mathcal{H}_j} = 1, \tag{2.4.3}$$

$$\lambda_n^4(i\lambda_n I - \mathcal{A}_j)U^n = (f_1^n, f_2^n, f_3^n, g_1^n, g_2^n, g_3^n, g_4^n) \longrightarrow 0 \quad \text{in} \quad \mathcal{H}_j, \quad j = 1, 2. \tag{2.4.4}$$

Equation (2.4.4) could be written as

$$i\lambda_n\varphi^n - u^n = \frac{f_1^n}{\lambda_n^4} \quad (2.4.5)$$

$$i\lambda_n\psi^n - v^n = \frac{f_2^n}{\lambda_n^4} \quad (2.4.6)$$

$$i\lambda_n\omega^n - z^n = \frac{f_3^n}{\lambda_n^4} \quad (2.4.7)$$

$$\lambda_n^2\varphi^n + \frac{\kappa}{\rho_1}(\varphi_{xx}^n + \psi_x^n + l\omega_x^n) + \frac{\kappa_0 l}{\rho_1}(\omega_x^n - l\varphi^n) = -\frac{g_1^n + i\lambda_n f_1^n}{\lambda_n^4}, \quad (2.4.8)$$

$$\lambda_n^2\psi^n + \frac{b}{\rho_2}\psi_{xx}^n - \frac{\kappa}{\rho_2}(\varphi_x^n + \psi^n + l\omega^n) - \frac{1}{\rho_2}\alpha(x)\theta_x^n = -\frac{g_2^n + i\lambda_n f_2^n}{\lambda_n^4}, \quad (2.4.9)$$

$$\lambda_n^2\omega^n + \frac{\kappa_0}{\rho_1}(\omega_{xx}^n - l\varphi_x^n) - \frac{\kappa l}{\rho_1}(\varphi_x^n + \psi^n + l\omega^n) = -\frac{g_3^n + i\lambda_n f_3^n}{\lambda_n^4} \quad (2.4.10)$$

$$i\lambda_n\theta^n - \frac{1}{\rho_3}\theta_{xx}^n + i\frac{T_0}{\rho_3}\lambda_n(\alpha\psi^n)_x = \frac{g_4^n + T_0\rho_3^{-1}(\alpha f_2^n)_x}{\lambda_n^4}. \quad (2.4.11)$$

Our goal is, using a multiplier method, to prove that $\|U^n\|_{\mathcal{H}_j} = o(1)$, this contradicts equation (2.4.3). We will establish the proof by several Lemmas. For simplicity, here and after we drop the index n.

Using equations (2.4.3), (2.4.5), (2.4.6), (2.4.7), (2.4.8), (2.4.9) and (2.4.10) we deduce that

$$\|\varphi_x\| = O(1), \quad \|\varphi\| = \frac{O(1)}{\lambda} \text{ and } \|\varphi_{xx}\| = O(\lambda).$$

$$\|\psi_x\| = O(1), \quad \|\psi\| = \frac{O(1)}{\lambda} \text{ and } \|\psi_{xx}\| = O(\lambda).$$

$$\|\omega_x\| = O(1), \quad \|\omega\| = \frac{O(1)}{\lambda} \text{ and } \|\omega_{xx}\| = O(\lambda).$$

Lemma 2.4.2. *(The dissipation)*

Under the above notations we have

$$\int_0^L |\theta_x|^2 dx = \frac{o(1)}{\lambda^4} \quad \text{and} \quad \int_0^L |\theta|^2 dx = \frac{o(1)}{\lambda^4}. \quad (2.4.12)$$

Proof. Multiplying equation (2.4.4) by the uniformly bounded sequence $U = (\varphi, \psi, \omega, u, v, z, \theta)$, we get

$$\int_0^L |\theta_x|^2 dx = -\mathrm{Re}((i\lambda - \mathcal{A}_j)U, U)_{\mathcal{H}_j} = \frac{o(1)}{\lambda^4}. \quad (2.4.13)$$

Finally, using Poincaré inequality, it follows the second asymptotic equality. □

Lemma 2.4.3. *(First information on ψ and ψ_x)*
Under the above notations, we have

$$\int_0^L \eta|\psi|^2 = \frac{o(1)}{\lambda^4} \quad \text{and} \quad \int_0^L \eta|\psi_x|^2 = \frac{o(1)}{\lambda^3} \tag{2.4.14}$$

where η is the function defined in Theorem 3.1

Proof. (i) We start by multiplying equation (2.4.11) by $\eta\bar{\psi}_x$, we get

$$\begin{aligned} T_0\int_0^L \eta\alpha|\psi_x|^2 &= \frac{T_0}{2}\int_0^L (\eta\alpha')'|\psi|^2 \\ +Re\left\{\rho_3\int_0^L(\eta'\theta+\eta\theta_x)\bar{\psi} + i\int_0^L \theta_x\lambda^{-1}\eta\bar{\psi}_{xx} + \frac{i}{\lambda}\int_0^L \eta'\theta_x\bar{\psi}_x\right\} &+ \frac{o(1)}{\lambda^5}. \end{aligned} \tag{2.4.15}$$

Using equation (2.4.12) and the fact that $\|\psi\| = \frac{O(1)}{\lambda}$, $\|\psi_x\| = O(1)$ and $\|\eta\psi_{xx}\| = O(\lambda)$ in (2.4.15), we get

$$\int_0^L \eta|\psi_x|^2 = o(1). \tag{2.4.16}$$

Next, multiplying equation (2.4.9) by $\eta\bar{\psi}$, we get

$$\begin{aligned} \rho_2\int_0^L \eta|\lambda\psi|^2 = b\int_0^L \eta|\psi_x|^2 + b\int_0^L \eta'\psi_x\bar{\psi} + \int_0^1 [\kappa(\psi+l\omega)+\alpha\theta_x]\eta\bar{\psi} \\ -\int_0^1 \kappa(\eta'\varphi\psi+\eta\varphi\psi_x) + \frac{o(1)}{\lambda^4}. \end{aligned} \tag{2.4.17}$$

Using equation (2.4.12), (2.4.16) and the fact that $\|\psi\| = \frac{O(1)}{\lambda}$ and $\|\omega\| = \frac{O(1)}{\lambda}$ in equation (2.4.17), we get

$$\int_0^L \eta|\psi|^2 = \frac{o(1)}{\lambda^2}. \tag{2.4.18}$$

(ii) Multiplying equation (2.4.15) by λ^2 and using (2.4.12), (2.4.18) and the fact that $\|\psi_{xx}\| = O(\lambda)$, we get

$$\int_0^L \eta|\psi_x|^2 = \frac{o(1)}{\lambda^2}. \tag{2.4.19}$$

(iii) Multiplying equation (2.4.17) by λ^2 and using (2.4.12), (2.4.18), (2.4.19) and the fact that $\|\lambda\omega\| = O(1)$, $\|\lambda\varphi\| = O(1)$, we obtain

$$\int_0^L \eta|\psi|^2 = \frac{o(1)}{\lambda^4}. \tag{2.4.20}$$

In addition, using (2.4.12), (2.4.20) and the fact that $\|\omega\| = \frac{O(1)}{\lambda}$, $\|\varphi_x\| = O(1)$ in (2.4.9), we get

$$\int_0^L |\eta\psi_{xx}|^2 = O(1). \tag{2.4.21}$$

Finally, multiplying equation (2.4.15) by λ^3, and using (2.4.20), (2.4.21) we deduce the second asymptotic behavior equation in (2.4.14). □

Lemma 2.4.4. *(Relation between φ and ψ)*
Let $\frac{1}{2} \le \gamma \le 1$. Under the above notations, assume that

$$\int_0^L \eta|\psi_x|^2 = \frac{o(1)}{\lambda^{2+2\gamma}}. \tag{2.4.22}$$

Then we have

$$\int_0^L \eta|\varphi_x|^2 = \frac{o(1)}{\lambda^{2\gamma}} \quad \text{and} \quad \int_0^L \eta|\varphi|^2 = \frac{o(1)}{\lambda^{2+2\gamma}}. \tag{2.4.23}$$

Proof. Let $l_N = \displaystyle\sum_{k=0}^{N} \frac{1}{2^k}$, we will prove by induction on $N \in \mathbb{N}$ that

$$\int_0^L \eta|\varphi_x|^2 = \frac{o(1)}{\lambda^{\gamma l_N}}. \tag{2.4.24}$$

(i) **Verification for** $N = 0$**.** Multiplying (2.4.9) by $\eta\bar{\varphi}_x$ and integrating over $]0, L[$, we get

$$\begin{aligned} \kappa\int_0^L \eta|\varphi_x|^2 &= -\rho_2\int_0^L \lambda^2(\eta\psi^n)_x\bar{\varphi} - b\int_0^L \lambda\eta\psi_x\lambda^{-1}\bar{\varphi}_{xx} \\ &- \int_0^L (\kappa\psi + \kappa l\omega + \alpha\theta_x)\eta\bar{\varphi}_x - b\int_0^L \psi_x\eta'\bar{\varphi}_x + \frac{o(1)}{\lambda^4} \end{aligned} \tag{2.4.25}$$

Using equations (2.4.12), (2.4.14) and the fact that $\|\varphi_{xx}\| = O(\lambda)$, $\|\varphi_x\| = O(1)$, $\|\varphi\| = \frac{O(1)}{\lambda}$ and $\|\omega\| = \frac{O(1)}{\lambda}$) in (2.4.25), we get

$$\int_0^L \eta|\varphi_x|^2 = o(1). \tag{2.4.26}$$

Now, multiplying equation (2.4.25) by λ^γ. Since $\gamma \leq 1$, then $\|\lambda^\gamma\omega\| = O(1)$ and $\|\lambda^\gamma\varphi\| = O(1)$. Using equations (2.4.12), (2.4.14), (2.4.22), (2.4.26) and the fact that $\|\varphi_{xx}\| = O(\lambda)$, we get

$$\int_0^L \eta|\varphi_x|^2 = \frac{o(1)}{\lambda^\gamma}. \tag{2.4.27}$$

Hence, the asymptotic behavior formula (2.4.24) is true for $N = 0$.

(ii) **Information on φ.** In addition, multiplying equation (2.4.8) by $\eta\bar{\varphi}$ and integrating over $]0, L[$, we get

$$\begin{aligned}\rho_1 \int_0^L \eta|\lambda\varphi|^2 &= \kappa \int_0^L (\eta|\varphi_x|^2 + (\eta'\varphi_x - \eta\psi_x)\bar{\varphi}) \\ +l \int_0^L &(\kappa + \kappa_0)\omega(\eta\bar{\varphi})_x + l^2\kappa_0 \int_0^L \eta|\varphi|^2 + \frac{o(1)}{\lambda^4}.\end{aligned} \tag{2.4.28}$$

Multiplying equation (2.4.28) by λ^γ. Then, using equation (2.4.27) and the fact that $\|\lambda^\gamma\omega\| = O(1)$, we get

$$\int_0^L \eta|\varphi|^2 = \frac{o(1)}{\lambda^{2+\gamma}}. \tag{2.4.29}$$

(iii) **Induction.** Suppose that the asymptotic behavior formula (2.4.24) is true for the order $N - 1$, then we have

$$\int_0^L \eta|\varphi_x|^2 = \frac{o(1)}{\lambda^{\gamma l_{N-1}}}. \tag{2.4.30}$$

Now, multiplying equation (2.4.28) by $\lambda^{\gamma l_{N-1}}$. Since $\gamma l_{N-1} \leq 2$, then $\|\lambda^{\frac{\gamma}{2} l_{N-1}}\omega\| = O(1)$. This implies that, using equations (2.4.14), (2.4.29), (2.4.30) and the fact that $\|\varphi_{xx}\| = O(\lambda)$, we get

$$\int_0^L \eta|\varphi|^2 = \frac{o(1)}{\lambda^{2+\gamma l_{N-1}}}. \tag{2.4.31}$$

On the other hand, using equation (2.4.31) and the fact that $\|\omega_x\| = O(1)$ in equation (2.4.8), we get

$$\int_0^L \eta|\varphi_{xx}|^2 = O(\lambda^{1-\frac{\gamma}{2}l_{N-1}}). \tag{2.4.32}$$

Noting that $\gamma+\frac{\gamma}{2}l_{N-1} = \gamma l_N$ and multiplying equation (2.4.25) by $\lambda^{\gamma+\frac{\gamma}{2}l_{N-1}}$. Then, using (2.4.14), (2.4.22), (2.4.30), (2.4.31), (2.4.32), we get

$$\int_0^L \eta|\varphi_x|^2 = \frac{o(1)}{\lambda^{\gamma l_N}}.$$

By consequences, the asymptotic behavior equation (2.4.24) is true for all $N \geq 0$.
(iv) **Result on φ_x.** Since $\lim_{N\to+\infty} l_N = \sum_{k=0}^{+\infty} \frac{1}{2^k} = 2$, we deduce the first desired asymptotic behavior equation:

$$\int_0^L \eta|\varphi_x|^2 = \frac{o(1)}{\lambda^{2\gamma}}. \tag{2.4.33}$$

(v) **Result on φ.** Multiplying equation (2.4.28) by $\lambda^{2\gamma}$. Then, using equations (2.4.29), (2.4.33) and the fact that $\|\lambda\omega\| = O(1)$, we deduce the second desired asymptotic behavior equation in (2.4.23). The proof is thus completed. □

Lemma 2.4.5. *(Relation between ψ and ψ_x)*
Let $\frac{1}{2} \leq \gamma \leq 1$. Under the above notations, assume that

$$\int_0^L \eta|\psi_x|^2 = \frac{o(1)}{\lambda^{2+2\gamma}}. \tag{2.4.34}$$

Then we have

$$\int_0^L \eta|\psi|^2 = \frac{o(1)}{\lambda^{4+2\gamma}}. \tag{2.4.35}$$

Proof. Let $l_N = \sum_{k=0}^{N} \frac{1}{2^k}$, we will prove by induction on $N \in \mathbb{N}$ that

$$\int_0^L \eta|\psi|^2 = \frac{o(1)}{\lambda^{4+\gamma l_N}}. \tag{2.4.36}$$

(i) **Verification for** $N = 0$**.** Multiplying equation (2.4.17) by $\lambda^{2+\gamma}$. Then, using equations (2.4.14), (2.4.34), Lemma 2.4.4 and the fact that $\|\omega\| = \frac{O(1)}{\lambda}$, we get

$$\int_0^L \eta|\psi|^2 = \frac{o(1)}{\lambda^{4+\gamma}}. \tag{2.4.37}$$

Hence, the asymptotic behavior formula (2.4.36) is true for $N = 0$.

(ii) **Induction.** Suppose that the asymptotic behavior formula (2.4.36) is true for the order $N-1$, then we have

$$\int_0^L \eta|\psi|^2 = \frac{o(1)}{\lambda^{4+\gamma l_{N-1}}}. \tag{2.4.38}$$

Multiplying equation (2.4.17) by $\lambda^{2+(\gamma+\frac{\gamma}{2}l_{N-1})}$. Since $\gamma+\frac{\gamma}{2}l_{N-1} \leq 2+2\gamma$ and $\gamma \leq 1$, then using equations (2.4.12), (2.4.34), (2.4.38), Lemma 2.4.4 , and the fact that $\|\lambda\omega\| = O(1)$, we get

$$\int_0^L \eta|\psi|^2 = \frac{o(1)}{\lambda^{4+(\gamma+\frac{\gamma}{2}l_{N-1})}}. \tag{2.4.39}$$

Since $\gamma + \frac{\gamma}{2}l_{N-1} = \gamma l_N$, we deduce the asymptotic behavior formula (2.4.35).

(iii) **Result.** Using the fact that $\lim_{N\to+\infty} l_N = \sum_{k=0}^{+\infty} \frac{1}{2^k} = 2$, we deduce the asymptotic behavior result (2.4.35). □

Lemma 2.4.6. *(Final information on* ψ *and* ψ_x*)*
Under the above notations, we have

$$\int_0^L \eta|\psi_x|^2 = \frac{o(1)}{\lambda^4} \quad \text{and} \quad \int_0^L \eta|\psi|^2 = \frac{o(1)}{\lambda^6}. \tag{2.4.40}$$

Proof. Let $\hat{l}_N = \sum_{k=1}^{N} \frac{1}{2^k}$. We will prove by induction on $N \in \mathbb{N}^\star$, that

$$\int_0^L \eta|\psi_x|^2 = \frac{o(1)}{\lambda^{2+2\hat{l}_N}}. \tag{2.4.41}$$

(i) **Verification for** $N = 1$**.** Using Lemma 2.4.3 we deduce that the asymptotic behavior equality (2.4.41) is true for $N = 1$.

(ii) **Induction.** Suppose that the asymptotic behavior equality (2.4.41) is true for $N-1$, then we have

$$\int_0^L \eta|\psi_x|^2 = \frac{o(1)}{\lambda^{2+2\hat{l}_{N-1}}}. \tag{2.4.42}$$

Then, applying Lemma 2.4.4 and Lemma 2.4.5 with $\gamma = \hat{l}_{N-1}$, we get

$$\int_0^L \eta|\varphi_x|^2 = \frac{o(1)}{\lambda^{2\hat{l}_{N-1}}} \quad \text{and} \quad \int_0^L \eta|\psi|^2 = \frac{o(1)}{\lambda^{4+2\hat{l}_{N-1}}}. \tag{2.4.43}$$

On the other hand, using equation (2.4.43) and the fact that $\hat{l}_{N-1} \leq 1$, $\|\lambda^{\hat{l}_{N-1}}\omega\| = O(1)$ in (2.4.9), we get

$$\|\psi_{xx}\| = \frac{O(1)}{\lambda^{\hat{l}_{N-1}}}. \tag{2.4.44}$$

Now, multiplying equation (2.4.15) by $\lambda^{3+\hat{l}_{N-1}}$. Then, using equations (2.4.12), (2.4.43) and (2.4.44), we get

$$\int_0^L \eta|\psi_x|^2 = \frac{o(1)}{\lambda^{3+\hat{l}_{N-1}}}. \tag{2.4.45}$$

Using the fact that $3+\hat{l}_{N-1} = 2+2\hat{l}_N$, we deduce the asymptotic behavior formula (2.4.41) for all $N \in \mathbb{N}^\star$.

(iii) **Result.** Using the fact that $\lim_{N\to+\infty} \hat{l}_{N-1} = \sum_{k=1}^{+\infty} \frac{1}{2^k} = 1$, we deduce the first asymptotic behavior equation in (2.4.40). Then applying Lemma 2.4.5 with $\gamma = 1$, we deduce the second asymptotic behavior equation in (2.4.40). The proof is thus completed. □

Lemma 2.4.7. *(Information on φ and φ_x)*

Under the above notations, we have

$$\int_0^L \eta|\varphi_x|^2 = \frac{o(1)}{\lambda^2} \quad \text{and} \quad \int_0^L \eta|\varphi|^2 = \frac{o(1)}{\lambda^4}. \tag{2.4.46}$$

Proof. Using Lemma 2.4.6 we deduce the asymptotic behavior equations (2.4.46) by applying Lemma 2.4.4 with $\gamma = 1$. the proof is thus completed. □

Lemma 2.4.8. *(Information on ω and ω_x)*
Under the above notations, we have

$$\int_0^L \eta|\omega_x|^2 = o(1) \quad \text{and} \quad \int_0^L \eta|\omega|^2 = \frac{o(1)}{\lambda^2}. \tag{2.4.47}$$

Proof. Multiply equation (2.4.8) by $\eta\overline{\omega_x}$. Then, using (2.4.40), (2.4.46) and the fact that $\|\omega_x\| = O(1)$, we get

$$(\kappa+\kappa_0)l\int_0^L \eta|\omega_x|^2 = \kappa\int_0^L \lambda\varphi_x\lambda^{-1}\eta\overline{\omega_{xx}} + o(1) \tag{2.4.48}$$

Then, using equation (2.4.46) and the fact that $\|\omega_{xx}\| = O(\lambda)$ in (2.4.48), we deduce the first asymptotic behavior equation in (2.4.47). Finally, multiplying equation (2.4.10) by $\eta\bar{\omega}$, we deduce the second asymptotic behavior equation in (2.4.47). The proof is thus completed. □

Proof of theorem 4.1 Using lemma 2.4.6, lemma 2.4.7 and lemma 2.4.8, we obtain $\|U\|_{\mathcal{H}_{ja_0,b_0}} = o(1)$. Then by applying lemma 2.3.4, we deduce that $\|U\|_{\mathcal{H}_j} = o(1)$, which is a contradiction with (2.4.3). This implies that $\|(i\lambda - \mathcal{A}_j)^{-1}\| = O(\lambda^4)$. This together with the fact that $i\mathbb{R} \subset \rho(\mathcal{A}_j)$ imply estimation (2.4.1) (see [6], [20]). The proof is thus completed.

Remark 2.4.9. *The conditions $\kappa \neq \kappa_0$ and $\frac{\rho_1}{\rho_2} \neq \frac{\kappa}{b}$ considered in Theorem 4.1 describe the natural physical problem. All other speed wave conditions have only mathematical sound. However, they do provide useful insight to the study of similar models arising from other applications.*

Remark 2.4.10. *In the case $\kappa = \kappa_0$ and $\frac{\kappa}{\rho_1} \neq \frac{b}{\rho_2}$, by a similar way used in Theorem 4.1, we can prove that*

$$E(t) \leq C\frac{1}{t}\|U_0\|^2_{D(\mathcal{A}_j)} \quad \forall t > 0. \tag{2.4.49}$$

Noting that, in this case, technically, the process of the proof is much easier to that of the natural general case of Theorem 4.1. In fact, we need to prove

$$\sup_{\lambda\in\mathbb{R}} \frac{1}{\lambda^2}\|(i\lambda I - \mathcal{A}_j)^{-1}\| < \infty.$$

From dissipation law we obtain

$$\int_0^L |\theta_x|^2 dx = \frac{o(1)}{\lambda^2} \text{ and } \int_0^L |\theta|^2 dx = \frac{o(1)}{\lambda^2}.$$

This leads to prove that

$$\int_0^L |\eta\psi_x|^2 dx = \frac{o(1)}{\lambda^2} \text{ and } \int_0^L |\eta\psi|^2 dx = \frac{o(1)}{\lambda^4}.$$

This implies that

$$\int_0^L |\eta\varphi_x|^2 dx = o(1) \text{ and } \int_0^L |\eta\varphi_x|^2 dx = \frac{o(1)}{\lambda^2}.$$

Here, we can use the condition $\kappa = \kappa_0$ in order to obtain

$$\int_0^L |\eta\omega_x|^2 dx = o(1) \text{ and } \int_0^L |\eta\omega_x|^2 dx = \frac{o(1)}{\lambda^2}.$$

Chapter 3

The influence of the coefficients of a system of wave equations coupled by velocities on its indirect boundary stabilization

3.1 Introduction

In [3], Ammar-Khodja and Bader studied the simultaneous boundary stabilization of a system of two wave equations coupling through the velocity terms. The system is described by:

$$\begin{aligned} u_{tt} - u_{xx} + b(x)y_t &= 0 \quad \text{in } (0,1) \times (0,+\infty), & (3.1.1) \\ y_{tt} - ay_{xx} - b(x)u_t &= 0 \quad \text{in } (0,1) \times (0,+\infty), & (3.1.2) \\ y_t(0,t) - a\left(y_x(0,t) + u_t(0,t)\right) &= 0 \quad \text{in } (0,+\infty), & (3.1.3) \\ u_x(0,t) - ay_t(0,t) &= 0 \quad \text{in } (0,+\infty), & (3.1.4) \\ u(1,t) = y(1,t) &= 0 \quad \text{in } (0,+\infty), & (3.1.5) \end{aligned}$$

where $a > 0$, $\alpha > 0$ are constants and $b \in C^0([0,1])$. Under the equal speed wave propagation condition *i.e.* $a = 1$, the authors proved that, system (3.1.1)-(3.1.5) is uniformly stable if and only if it is strongly stable and the coupling parameter b verifies that $\bar{b} := \int_0^1 b(x)dx \neq (2k+1)\frac{\pi}{2}$ for any $k \in \mathbb{Z}$. Moreover, when $a \neq 1$, they proved that system (3.1.1)-(3.1.5) is uniformly stable if and only if it is strongly stable and there exist $p, q \in \mathbb{Z}$ such that $a = \frac{(2p+1)^2}{q^2}$. Noting that, the above system is directly damped by two related boundary controls.

Moreover, in [33], Toufayli considered a multidimensional system of coupled wave equations subject to one boundary feedback described by

$$\begin{aligned} u_{tt} - \Delta u + by_t &= 0 \quad \text{in } \Omega \times (0,+\infty), & (3.1.6) \\ y_{tt} - a\Delta y - bu_t &= 0 \quad \text{in } \Omega \times (0,+\infty), & (3.1.7) \\ \partial_\nu y - y_t &= 0 \quad \text{on } \Gamma_1 \times (0,+\infty), & (3.1.8) \\ u &= 0 \quad \text{on } \Gamma \times (0,+\infty), & (3.1.9) \\ y &= 0 \quad \text{on } \Gamma_0 \times (0,+\infty), & (3.1.10) \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ is an open bounded domain of class C^2, $\partial\Omega = \Gamma_0 \cup \Gamma_1$, with $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, ν is the unit normal vector to Γ_1 pointing toward the exterior of Ω, $a > 0$ and $b \in \mathbb{R}^\star$ are constants. Under the equal speed wave propagation condition (in the case $a = 1$) and if the coupling parameter b is small enough, she established an exponential energy decay estimate. However, on the contrary ($a \neq 1$ or for large b), no stability type has been discussed. We think that the conditions on a and b are technical and could be improved. Then the influence of the arithmetic property of the ratio of the wave propagation speeds a and of the algebraic property of the coupling parameter b on the stability of the system of two coupled wave equations when only one of these equation is effectively damped remains an open problem. Our objective in this chapter is to give a complete answer of this interesting open problem in the one dimensional case. The aim of this chapter is to investigate the energy decay rate of a coupled wave equations damped by one boundary feedback. The system is described by:

$$u_{tt} - u_{xx} + by_t = 0 \quad \text{in } (0,1) \times (0,+\infty), \tag{3.1.11}$$

$$y_{tt} - ay_{xx} - bu_t = 0 \quad \text{in } (0,1) \times (0,+\infty), \tag{3.1.12}$$

$$y_x(0,t) - y_t(0,t) = 0 \quad \text{in } (0,+\infty), \tag{3.1.13}$$

$$u(1,t) = y(1,t) = u(0,t) = 0 \quad \text{in } (0,+\infty), \tag{3.1.14}$$

where $a > 0$ and $b \in \mathbb{R}^\star$ are constants. First, we prove that system (3.1.11)-(3.1.14) is strongly stable if and only if the coupling parameter b is outside a well determined discrete set S_s of exceptional values. Consequently, the strong stability does not hold in general. Next, for $b \notin S_s$, we show that the energy decay rate of system (3.1.11)-(3.1.14) is greatly influenced by the nature of the coupling parameter b (an additional condition on b) and by the arithmetic property of the ratio of the wave propagation speeds a. Indeed, in the case of $a = 1$ when the waves propagate at the same speed and if there exist no $k \in \mathbb{Z}$ such that $b = k\pi$, we establish an exponential stability of system (3.1.11)-(3.1.14). Otherwise, we prove that the above conditions on a and b for the exponential stability of the system are optimal in the sense that the absence of one of them turns system (3.1.11)-(3.1.14) to be not exponentially sable. In this case, it is natural to expect a polynomial energy decay rate also depending on the nature of a and b. Roughly speaking, if $a = 1$

and b is of the form $k\pi$ for $k \in \mathbb{Z}$, an optimal energy decay rate of type $\frac{1}{\sqrt{t}}$ is established. Furthermore, in the case $a \neq 1$, if $a \in \mathbb{Q}$ and b small enough **or** if $\sqrt{a} \in \mathbb{Q}$, we obtain a polynomial energy decay rate of type $\frac{1}{\sqrt{t}}$. The frequency domain approach combined with a multiplier method is used.

The notion of indirect damping mechanisms has been introduced by Russell in [30], and since this time, it retains the attention of many authors. In particular, the boundary stabilization of the system of two wave equations coupled through the zero order terms has been studied with different approaches. In [1], Alabau-Boussouira studied the boundary indirect stabilization of a system of two second order evolution equations coupled through the zero order terms. The lack of uniform stability was proved by a compact perturbation argument and a polynomial energy decay rate of type $\frac{1}{\sqrt{t}}$ is obtained by a general integral inequality in the case where the waves propagate at the same speed and Ω is a star-shaped domain in $\mathbb{R}^N$, or in the case where the ratio of the wave propagation speeds of the two equations is equal $1/k^2$ with k being an integer and Ω is a cubic domain of $\mathbb{R}^3$. Liu and Rao in [21] considered a system of two coupled wave equations with one boundary damping and they proved that the energy of the system decays at the rate $\frac{1}{t}$ for smooth initial data on a N-dimensional domain Ω with usual geometrical condition when the waves propagate at the same speed. On the contrary, under some arithmetic condition on the ratio of the wave propagation speeds of the two equations, they established a polynomial energy decay rate for smooth initial data on a one-dimensional domain. Ammari and Mehrenberger in [4], gave a characterization of the stability of a system of two evolution equations coupling through the velocity terms subject to one bounded viscous feedback damping. Note nevertheless that our system does not enter in the framework of the cited papers.

This chapter is organized as follows: In section 2, using semigroup theory, we prove the well-posedness of the problem while using the decomposition spectral theory, we establish the strong stability of the system if and only if b is outside a discrete set of exceptional values. The section 3 is devoted to study the exponential stability of system (3.1.11)-(3.1.14) when $a = 1$ and b is not of the form $k\pi$, for integer k. The frequency domain approach combined with a multiplier method is used. In Section 4, first, using a spectrum method, we show that the condition $b \neq 2k\pi$ for $k \in \mathbb{Z}$ is necessary to obtain the exponential stability of the system. We next establish an optimal polynomial energy decay rate of type $\frac{1}{\sqrt{t}}$ using a frequency domain approach. Section 5 is devoted to study the polynomial energy decay rate in the case $a \neq 1$. Indeed, if $a \in \mathbb{Q}$ and b small enough **or** $\sqrt{a} \in \mathbb{Q}$ we establish a polynomial energy decay rate of type $\frac{1}{\sqrt{t}}$. The frequency domain approach combined with a multiplier method is used.

3.2 Abstract setting and strong stability

In this section, we study existence, uniqueness and strong stability of system (3.1.11)-(3.1.14). We start by studying the well-posedness of the problem.

3.2.1 Semigroup solution

This subsection is devoted for the study of existence, uniqueness of solution and regularity of solution of system (3.1.11)-(3.1.14). Now we define the space

$$H_R^1(0,1) = \left\{y \in H^1(0,1) : y(1) = 0\right\}$$

and the energy space

$$\mathcal{H} = H_0^1(0,1) \times L^2(0,1) \times H_R^1(0,1) \times L^2(0,1),$$

which is endowed with the inner product

$$\left(U,\widetilde{U}\right)_{\mathcal{H}}=\int_0^1\left(u_x\overline{\widetilde{u}_x}+v\overline{\widetilde{v}}+ay_x\overline{\widetilde{y}_x}+z\overline{\widetilde{z}}\right)dx,\quad \forall U=(u,v,y,z),\widetilde{U}=(\widetilde{u},\widetilde{v},\widetilde{y},\widetilde{z})\in\mathcal{H}.$$

We next define the unbounded linear operator $\mathcal{A}$: $D(\mathcal{A})\subseteq\mathcal{H}\longrightarrow\mathcal{H}$, by

$$\begin{aligned}D(\mathcal{A})=&\big\{U=(u,v,y,z)\in\mathcal{H}:u,y\in H^2,v\in H_0^1,\\&z\in H_R^1\text{ and }y_x(0)=z(0)\big\}\end{aligned}\tag{3.2.1}$$

and

$$\mathcal{A}U=(v,u_{xx}-bz,z,ay_{xx}+bv),\ \forall\ U=(u,v,y,z)\in D(\mathcal{A}).\tag{3.2.2}$$

If $U=(u,u_t,y,y_t)$ is a regular solution of system (3.1.11)-(3.1.14), then we rewrite this system as the following evolutionary equation

$$\begin{cases}U'(t)&=&\mathcal{A}U(t),\\U(0)&=&U_0\in\mathcal{H}.\end{cases}\tag{3.2.3}$$

Now, we can state the following proposition that aims to show that $\mathcal{A}$ generates a C_0- semigroup of contractions.

Proposition 3.2.1. *The operator $\mathcal{A}$ is m-dissipative in the energy space $\mathcal{H}$. In addition, the linear bounded operator $\mathcal{A}^{-1}$ is compact in $\mathcal{H}$.*

Proof. Firstly, for all $U=(u,v,y,z)\in D(\mathcal{A})$, a direct computation gives that

$$Re(\mathcal{A}(u,v,y,z),(u,v,y,z))_{\mathcal{H}}=-a|z(0)|^2\leq 0.\tag{3.2.4}$$

Which implies that $\mathcal{A}$ is dissipative in the energy space $\mathcal{H}$.
Next, for any given $F=(f,g,h,k)\in\mathcal{H}$, we solve the equation

$$\mathcal{A}U=F.\tag{3.2.5}$$

Then, we consider the following systems

$$\begin{cases} u_{xx} = g + bh, \text{ in } L^2(0,1), \\ u(0) = u(1) = 0 \end{cases} \tag{3.2.6}$$

and

$$\begin{cases} ay_{xx} = k - bf, \text{ in } L^2(0,1), \\ y(1) = 0, y_x(0) = h(0). \end{cases} \tag{3.2.7}$$

It is easy to see that equation (3.2.6) admits a unique solution $u \in H_0^1 \cap H^2$ and equation (3.2.7) admits a unique solution $y \in H_R^1 \cap H^2$. Next, define $v = f$ and $z = h$, then $U = (u, v, y, z) \in D(\mathcal{A})$ is the unique solution of equation (3.2.5). Thus the operator $\mathcal{A}$ is invertible, that is $0 \in \rho(\mathcal{A})$. Then by the contraction principle, we easily get $R(\lambda I - \mathcal{A}) = \mathcal{H}$ for sufficient small $\lambda > 0$. This, together with the dissipativeness of $\mathcal{A}$, imply the density of $D(\mathcal{A})$ in $\mathcal{H}$ (see [27], Theorem 1.4.6). This implies that $\mathcal{A}$ is m-dissipative.
Finally, the Sobolev embedding theorem asserts that $\mathcal{A}^{-1}$ is a compact operator. Thus the proof is complete. □

Now, Thanks to Lumer Phillips theorem (see [27], Theorem 1.4.3), the operator $\mathcal{A}$ generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ on the energy space $\mathcal{H}$. Then, we have the following existence and uniqueness result

Theorem 3.2.2. *For all $U_0 \in \mathcal{H}$ there exists a unique $U \in C^0([0;+\infty);\mathcal{H})$ weak solution of the Cauchy problem (3.2.3). Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$U \in C^1([0;+\infty);\mathcal{H}) \cap C^0([0;+\infty);D(\mathcal{A}))$$

is the strong solution of the Cauchy problem (3.2.3).

3.2.2 Strong stability

In this part, we prove that the strong stability of the system (3.1.11)-(3.1.14) is greatly influenced by the nature of the coupling parameter b. This statement is subject of the following theorem.

Theorem 3.2.3. *The semigroup of contractions $e^{t\mathcal{A}}$ is strongly stable on the energy space $\mathcal{H}$ in the sense that $\lim_{t\to+\infty} ||e^{t\mathcal{A}}U_0||_{\mathcal{H}} = 0$ for all $U_0 \in \mathcal{H}$ if and only if*

$$b^2 \neq \frac{(k_1^2 - ak_2^2)(ak_1^2 - k_2^2)\pi^2}{(a+1)(k_1^2+k_2^2)}, \quad \forall k_1, k_2 \in \mathbb{Z}. \qquad \text{(SC1)}$$

Proof. Since the operator $\mathcal{A}$ has a compact resolvent in the energy space $\mathcal{H}$, then using spectral decomposition theory (see [7]), system (3.1.11)-(3.1.14) is strongly stable if $\mathcal{A}$ has no pure imaginary eigenvalues. Since $0 \in \rho(\mathcal{A})$, we only need to check that $\text{Ker}(i\lambda I - \mathcal{A}) = \{0\}$ for all real number $\lambda \neq 0$. Then, let $\lambda \in \mathbb{R}$; $\lambda \neq 0$ and $U = (u, v, y, z) \in D(\mathcal{A})$ such that

$$\mathcal{A}U = i\lambda U. \tag{3.2.8}$$

Using (3.2.4) and (3.2.8), we get

$$-a|z(0)|^2 = Re(\mathcal{A}U, U)_{\mathcal{H}} = 0.$$

This implies that

$$y_x(0) = z(0) = 0.$$

Now, detailing equation (3.2.8), we obtain $v = i\lambda u$, $z = i\lambda y$ and the following system:

$$\lambda^2 u + u_{xx} - i\lambda b y = 0, \tag{3.2.9}$$

$$\lambda^2 y + a y_{xx} + i\lambda b u = 0, \tag{3.2.10}$$

$$u(1) = y(1) = u(0) = y(0) = y_x(0) = 0. \tag{3.2.11}$$

Combining (3.2.9) - (3.2.10) and (3.2.11), we get

$$
\begin{aligned}
au_{xxxx} + \lambda^2(a+1)u_{xx} + \lambda^2(\lambda^2 - b^2)u &= 0, && (3.2.12)\\
u(0) = u_{xx}(0) &= 0, && (3.2.13)\\
u(1) = u_{xx}(1) &= 0, && (3.2.14)\\
u_{xxx}(0) + \lambda^2 u_x(0) &= 0. && (3.2.15)
\end{aligned}
$$

The solution u of equation (3.2.12) is given by $u(x) = \sum_{j=1}^{4} c_j e^{r_j x}$, where

$$
r_1 = \sqrt{\frac{-\lambda^2(a+1) - \lambda\sqrt{\lambda^2(a-1)^2 + 4ab^2}}{2a}}, \quad r_2 = -r_1, \tag{3.2.16}
$$

$$
r_3 = \sqrt{\frac{-\lambda^2(a+1) + \lambda\sqrt{\lambda^2(a-1)^2 + 4ab^2}}{2a}}, \quad r_4 = -r_3, \tag{3.2.17}
$$

and $c_j \in \mathbb{C}$ are constants. If $\lambda = \pm b$, then system (3.2.12)-(3.2.15) admits only the trivial solution. Our goal is to find a non trivial solution of system (3.2.12)-(3.2.15), then assume that $\lambda \neq \pm b$. Since $r_1^2 - r_3^2 \neq 0$, then, using boundary conditions (3.2.13), we get

$$
u(x) = 2c_1 \sinh(r_1 x) + 2c_3 \sinh(r_3 x).
$$

From the boundary conditions (3.2.14), we distinguish the following four cases

Case 1. $\sinh(r_1) \neq 0$ and $\sinh(r_3) \neq 0$. Since $r_1^2 - r_3^2 \neq 0$, then using boundary conditions (3.2.14), it is easy to see that $u(x) = 0$. Consequently $U = 0$.

Case 2. $\sinh(r_1) \neq 0$ and $\sinh(r_3) = 0$. Using boundary conditions (3.2.14) we deduce $c_1 = 0$ and then $u(x) = 2c_3 \sinh(r_1 x)$. Finally, using boundary condition (3.2.15), we get $u(x) = 0$. Consequently $U = 0$.

Case 3. $\sinh(r_1) = 0$ and $\sinh(r_3) \neq 0$. By a same argument as in case 2., we get $u(x) = 0$ and consequently $U = 0$.

Case 4. $\sinh(r_1) = 0$ and $\sinh(r_3) = 0$. It follows that

$$
r_1 = ik_1\pi \quad \text{and} \quad r_3 = ik_2\pi, \quad \text{where } k_1, k_2 \in \mathbb{Z}. \tag{3.2.18}
$$

Inserting equation (3.2.18) into equations (3.2.16) and (3.2.17) respectively, we get

$$\lambda^2(a+1) + \lambda\sqrt{\lambda^2(a-1)^2 + 4ab^2} = 2ak_1^2\pi^2 \tag{3.2.19}$$

and

$$\lambda^2(a+1) - \lambda\sqrt{\lambda^2(a-1)^2 + 4ab^2} = 2ak_2^2\pi^2. \tag{3.2.20}$$

By adding equations (3.2.19) and (3.2.20), we obtain

$$\lambda^2 = \frac{a}{a+1}(k_1^2 + k_2^2)\pi^2. \tag{3.2.21}$$

Subtract equations (3.2.19) and (3.2.20), we obtain

$$\lambda\sqrt{\lambda^2(a-1)^2 + 4ab^2} = a(k_1^2 - k_2^2)\pi^2. \tag{3.2.22}$$

Inserting (3.2.21) into (3.2.22), we get

$$b^2 = \frac{(k_1^2 - ak_2^2)(ak_1^2 - k_2^2)}{(a+1)(k_1^2 + k_2^2)}\pi^2. \tag{3.2.23}$$

Hence, if (3.2.23) holds, then $i\lambda$, λ given in (3.2.21) is an eigenvalue of $\mathcal{A}$ with the corresponding eigenvector $U = (u, i\lambda u, \frac{-i}{\lambda b}(\lambda^2 u + u_{xx}), \frac{1}{b}(\lambda^2 u + u_{xx}))$ where u is given by

$$u(x) = 2i\frac{k_2(ak_1^2 - k_2^2)}{k_1(k_1^2 - ak_2^2)}\sin(k_1\pi x) + 2i\sin(k_2\pi x).$$

Conversely, if (3.2.23) does not hold, then $i\lambda$ is not an eigenvalue of $\mathcal{A}$ and the system (3.1.11)-(3.1.14) is strongly stable. The proof is thus complete. □

3.3 Exponential stability of the system in the case $a = 1$ and $b \neq k\pi$ for $k \in \mathbb{Z}$

In this section, under necessary and sufficient conditions on the coupling parameter b and the ratio of the wave propagation speeds a, we will establish the uniform stability of system (3.1.11)-(3.1.14). Note that, in the case $a = 1$, condition (SC1) is reduced to

$$b^2 \neq \frac{\pi}{\sqrt{2}} \frac{k_1^2 - k_2^2}{\sqrt{k_1^2 + k_2^2}} \quad \forall k_1, k_2 \in \mathbb{Z}. \qquad \text{(SC2)}$$

Theorem 3.3.1. *(Exponential decay rate) Assume that $a = 1$, b satisfies* (SC2) *and there is no $k \in \mathbb{Z}$ such that $b = k\pi$. Then there exist positive constants $M > 0$, $\omega > 0$ such that for all $(u_0, u_1, y_0, y_1) \in \mathcal{H}$ the energy of the system (3.1.11)-(3.1.14) satisfies the following exponential decay estimate:*

$$E(t) \leq M e^{-\omega t} E(0), \qquad \forall t > 0.$$

Proof. From a result of Huang [15] and Prüss [28], a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in a Hilbert space $\mathcal{H}$ is exponentially stable if and only if the condition (H1) and (H2) below are satisfied:

$$i\mathbb{R} \subset \rho(\mathcal{A}) \qquad \text{(H1)},$$

$$\sup_{\lambda \in \mathbb{R}} \|(i\lambda I - \mathcal{A})^{-1}\| < +\infty \qquad \text{(H2)}.$$

Since the resolvent of $\mathcal{A}$ is compact in the energy space and $0 \in \rho(\mathcal{A})$, then condition (H1) is satisfied by the fact that $\mathcal{A}$ has no pure imaginary eigenvalues (proved in Theorem 3.2.3).

We now prove (H2) by a contradiction argument. Suppose that (H2) does not hold. Then there exist two sequences $(\lambda_n) \subset \mathbb{R}$ and $(U^n) = (u^n, v^n, y^n, z^n) \subset D(\mathcal{A})$ such that

$$|\lambda_n| \longrightarrow +\infty, \tag{3.3.1}$$

$$\|U^n\|_{\mathcal{H}} = 1, \tag{3.3.2}$$

$$(i\lambda_n I - \mathcal{A})(u^n, v^n, y^n, z^n) = (f_1^n, g_1^n, f_2^n, g_2^n) \longrightarrow 0 \quad \text{in} \quad \mathcal{H}. \tag{3.3.3}$$

Detailing equation (3.3.3), we get

$$i\lambda_n u^n - v^n = f_1^n \longrightarrow 0 \quad \text{in} \quad H_0^1, \tag{3.3.4}$$

$$i\lambda_n v^n - u_{xx}^n + bz^n = g_1^n \longrightarrow 0 \quad \text{in} \quad L^2, \tag{3.3.5}$$

$$i\lambda_n y^n - z^n = f_2^n \longrightarrow 0 \quad \text{in} \quad H_R^1, \tag{3.3.6}$$

$$i\lambda_n z^n - y_{xx}^n - bv^n = g_2^n \longrightarrow 0 \quad \text{in} \quad L^2. \tag{3.3.7}$$

Our objective is to prove that $\|U^n\|_{\mathcal{H}} = o(1)$, this contradicts (3.3.2). The demonstration is divided into several steps.

Step 1. *(The dissipation)* Since the sequence (U^n) is uniformly bounded in $\mathcal{H}$, then using (3.2.4) and (3.3.3), we get

$$|z^n(0)|^2 = \text{Re}((i\lambda_n I - \mathcal{A})U^n, U^n)_{\mathcal{H}} = o(1). \tag{3.3.8}$$

It follows that

$$|y_x^n(0)| = o(1). \tag{3.3.9}$$

Combining (3.3.8) and (3.3.6), we get

$$|y^n(0)| = \frac{o(1)}{\lambda_n}. \tag{3.3.10}$$

On the other hand, from (3.3.2) we deduce that v^n and z^n are uniformly bounded in $L^2(0,1)$. Then, equations (3.3.4) and (3.3.6) give that

$$\|u^n\| = \frac{O(1)}{\lambda_n} \quad \text{and} \quad \|y^n\| = \frac{O(1)}{\lambda_n}. \tag{3.3.11}$$

Step 2. *(Multiplier method)* Here and after, for simplicity, we drop the index n. Substitute v, z in equations (3.3.5) and (3.3.7) by (3.3.4) and (3.3.6) respectively, we obtain the following system:

$$\lambda^2 u + u_{xx} - i\lambda b y = -g_1 - i\lambda f_1 - b f_2, \tag{3.3.12}$$

$$\lambda^2 y + y_{xx} + i\lambda b u = -g_2 - i\lambda f_2 + b f_1. \tag{3.3.13}$$

Let $h \in W^{1,\infty}(0,1)$, multiplying equation (3.3.12) by $2h\overline{u_x}$ and integrate by parts, we get

$$\begin{aligned} -\int_0^1 h'|\lambda u|^2 - \int_0^1 h'|u_x|^2 + h(1)|u_x(1)|^2 - h(0)|u_x(0)|^2 \\ -2\mathrm{Re}\{i\lambda b \int_0^1 y h \overline{u_x}\} = o(1). \end{aligned} \tag{3.3.14}$$

Note that, since f_1 converges to zero in $H_0^1(0,1)$ and λu is uniformly bounded in $L^2(0,1)$, then

$$\int_0^1 \lambda f_1 h u_x dx = -\int_0^1 \lambda u (f_1' h + h' f_1) dx = o(1).$$

Similarly, multiplying equation (3.3.13) by $2h\overline{y_x}$ and using (3.3.10), we obtain

$$-\int_0^1 h'|\lambda y|^2 - \int_0^1 h'|y_x|^2 + h(1)|y_x(1)|^2 + 2\mathrm{Re}\{i\lambda b \int_0^1 u h \overline{y_x}\} = o(1). \tag{3.3.15}$$

Combining (3.3.14) and (3.3.15), we obtain

$$\begin{aligned} -\int_0^1 h'|\lambda u|^2 - \int_0^1 h'|u_x|^2 - \int_0^1 h'|\lambda y|^2 - \int_0^1 h'|y_x|^2 + h(1)|u_x(1)|^2 \\ -h(0)|u_x(0)|^2 + h(1)|y_x(1)|^2 = o(1). \end{aligned} \tag{3.3.16}$$

By taking $h = 1$, we get

$$|y_x(1)|^2 + |u_x(1)|^2 - |u_x(0)|^2 = o(1). \tag{3.3.17}$$

Let us suppose that $u_x(0) = o(1)$, then using (3.3.17) we deduce that

$$u_x(1) = o(1) \text{ and } y_x(1) = o(1).$$

In this case, by taking $h = x$ in equation (3.3.16) we get

$$\int_0^1 |\lambda u|^2 + \int_0^1 |u_x|^2 + \int_0^1 |\lambda y|^2 + \int_0^1 |y_x|^2 = o(1). \tag{3.3.18}$$

Hence $\|U^n\|_{\mathcal{H}} = o(1)$, and we obtain the desired contradiction. Therefore, in order to complete the prove, we need to show that $u_x(0) = o(1)$.

Step 3. In this step we will prove that $u_x(0) = o(1)$. Let $Y = (u, u_x, y, y_x)^T$, then system (3.3.12)-(3.3.13) could be written as

$$Y_x = BY + G + \lambda F, \tag{3.3.19}$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\lambda^2 & 0 & i\lambda b & 0 \\ 0 & 0 & 0 & 1 \\ -i\lambda b & 0 & -\lambda^2 & 0 \end{pmatrix}, \quad F = (F_j) = \begin{pmatrix} 0 \\ -if_1 \\ 0 \\ -if_2 \end{pmatrix},$$

$$G = (G_j) = \begin{pmatrix} 0 \\ -g_1 - bf_2 \\ 0 \\ -g_2 + bf_1 \end{pmatrix}. \tag{3.3.20}$$

Using Ordinary Differential Equation Theory, the solution of equation (3.3.19) is given by

$$Y(x) = e^{Bx}Y_0 + \int_0^x e^{B(x-z)}G(z)dz + \int_0^x \lambda e^{B(x-z)}F(z)dz, \tag{3.3.21}$$

where, from (3.3.9)-(3.3.10), we have

$$Y_0 = (u(0), u_x(0), y(0), y_x(0))^T = \left(0, u_x(0), \frac{o(1)}{\lambda}, o(1)\right)^T.$$

Using Maple software, the exponential of the matrix B is given by

$$e^B = \begin{pmatrix} A_1 & 0 & iA_3 & 0 \\ \frac{b^2}{8}A_1 - \frac{b}{2}A_4 & A_1 & \frac{ib}{2}A_2 + \frac{ib^2}{8}A_3 & iA_3 \\ -iA_3 & 0 & A_1 & 0 \\ \frac{-ib}{2}A_2 - \frac{ib^2}{8}A_3 & -iA_3 & \frac{-b}{2}A_4 + \frac{b^2}{8}A_1 & A_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\lambda A_2 & 0 & i\lambda A_4 & 0 \\ 0 & 0 & 0 & 0 \\ -i\lambda A_4 & 0 & -\lambda A_2 & 0 \end{pmatrix} + (o(1))$$

where $(o(1)) = (o_{ij}) \in \mathfrak{M}_4(\mathbb{C})$, $o_{ij} = o(1)$ and

$$A_1 = \cos\lambda \cos\frac{b}{2},\ A_2 = \sin\lambda \cos\frac{b}{2},\ A_3 = \sin\lambda \sin\frac{b}{2},\ A_4 = \cos\lambda \sin\frac{b}{2}.$$

Since $G_1 = G_3 = 0$, and using the fact that $G_2 = -g_1 - bf_2$ and $G_4 = -g_2 + bf_1$ converge to zero in $L^2(0,1)$, we get

$$\int_0^x e^{B(x-z)}G(z)dz = o(1). \tag{3.3.22}$$

On the other hand, using integration by parts, the integral $\int_0^x \lambda e^{B(x-z)}F(z)dz$ could be written as

$$\begin{aligned} \int_0^x \lambda e^{B(x-z)}F(z)dz = &-\int_0^x \lambda B^{-1}e^{B(x-z)}F'(z)dz \\ &+\lambda B^{-1}F(x) - \lambda B^{-1}e^{Bx}F(0), \end{aligned} \tag{3.3.23}$$

where B^{-1}, the inverse matrix of B is given by

$$B^{-1} = \begin{pmatrix} 0 & \frac{1}{-\lambda^2+b^2} & 0 & \frac{ib}{\lambda(-\lambda^2+b^2)} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{-ib}{\lambda(-\lambda^2+b^2)} & 0 & \frac{1}{-\lambda^2+b^2} \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since $F_1 = F_3 = 0$, and using the fact that $F_2 = -if_1$ and $F_4 = -if_2$ converge to zero in H_0^1 and H_R^1 respectively, and following straightforward calculations in (3.3.23), we get

$$\int_0^x \lambda e^{B(x-z)}F(z)dz = o(1). \tag{3.3.24}$$

Substituting equations (3.3.22) and (3.3.24) in (3.3.21) and take $x = 1$, we obtain

$$Y(1) = e^{B} Y_0 + o(1). \tag{3.3.25}$$

Detailing (3.3.25), we obtain the following equations

$$y_x(1) = -i \sin\lambda \sin\frac{b}{2} u_x(0) + o(1), \tag{3.3.26}$$

$$u_x(1) = \cos\lambda \cos\frac{b}{2} u_x(0) + o(1). \tag{3.3.27}$$

Inserting equations (3.3.26) and (3.3.27) in (3.3.17), we obtain

$$(\sin^2\lambda \sin^2\frac{b}{2} + \cos^2\lambda \cos^2\frac{b}{2} - 1)|u_x(0)|^2 = o(1). \tag{3.3.28}$$

A direct calculation in (3.3.28) gives

$$(\cos^2\lambda \sin^2\frac{b}{2} + \sin^2\lambda \cos^2\frac{b}{2})|u_x(0)|^2 = o(1). \tag{3.3.29}$$

If $u_x(0)$ does not converge to zero in $\mathbb{C}$, then from (3.3.29), we deduce that

$$\cos^2\lambda \sin^2\frac{b}{2} = o(1) \qquad \text{and} \qquad \sin^2\lambda \cos^2\frac{b}{2} = o(1). \tag{3.3.30}$$

Since there is no $k \in \mathbb{Z}$, such that $b = k\pi$, then from (3.3.30), we get $1 = \cos^2\lambda + \sin^2\lambda = o(1)$ contradiction. Consequently, $u_x(0) = o(1)$ and the proof is complete. □

Remark 3.3.2. *In this remark we give a graphical interpretation of the exponential stability of system (3.1.11)-(3.1.14).*

Remark 3.3.3. *We will proof that the conditions of Theorem 3.3.1 are optimal in the sense that the lack of one of this conditions yield system (3.1.11)-(3.1.14) non exponentially stable.*

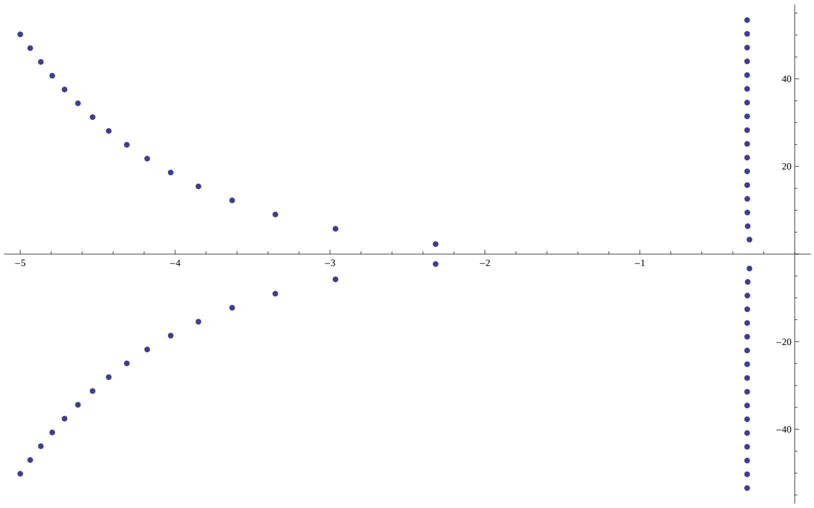

Figure 3.1: Eigenvalues in the case $a = 1$ and $b = 1$

3.4 Optimal polynomial decay rate in the case $a = 1$ and $b = k\pi$, for $k \in \mathbb{Z}$

In this section, we consider system (3.1.11)-(3.1.14) when the two waves propagate with same speed $a = 1$ and when the coupling parameter is on the form $b = k\pi$, $k \in \mathbb{Z}$. Firstly, we show that the condition $b \neq k\pi$, $k \in \mathbb{Z}$ imposed in Theorem 3.3.1 is a sufficient and necessary condition for the exponential stability of the system. We next establish an optimal polynomial energy decay estimate in the case $a = 1$ and $b = k\pi$, $k \in \mathbb{Z}$. We start by showing a general optimality result.

3.4.1 Optimality of a polynomial energy decay rate by spectral approach

In this subsection, we give a spectral approach for the optimality of a polynomial energy decay rate of C_0-semigroups.

Theorem 3.4.1. *Let $e^{t\mathcal{A}}$ be a C_0-semigroup of contractions generated by an operator $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. Let $(\lambda_{k,n})_{1\leq k\leq k_0, n\geq 1}$ denote the kth branch of eigenvalues of $\mathcal{A}$ and $(e_{k,n})_{1\leq k\leq k_0, n\geq 1}$ the system of normalized associated eigenvectors. Assume that for each $1 \leq k \leq k_0$ there exist a positive sequence $\mu_{k,n} \longrightarrow +\infty$ as $n \longrightarrow +\infty$ and two positive constants $\alpha_k > 0$, $\beta_k > 0$ such that*

$$\Re\lambda_{k,n} \sim -\frac{\beta_k}{\mu_{k,n}^{\alpha_k}} \text{ and } |\Im\lambda_{k,n}| \sim \mu_{k,n} \text{ as } n \longrightarrow +\infty. \tag{3.4.1}$$

Assume that, $i\mathbb{R} \subset \rho(\mathcal{A})$ and for any $u_0 \in D(\mathcal{A})$, there exists constant $M > 0$ independent of u_0 such that

$$\|e^{t\mathcal{A}}u_0\|_{\mathcal{H}} \leq \frac{M}{t^{\frac{1}{l_k}}}\|u_0\|_{D(\mathcal{A})}, \;\; l_k = \max_{1\leq k\leq k_0}(\alpha_k), \;\; \forall t > 0. \tag{3.4.2}$$

Then the decay rate (3.4.2) *is optimal in the sense that for any $\varepsilon > 0$, we cannot expect the decay rate $\dfrac{1}{t^{\frac{1}{l_k}+\varepsilon}}$*

Proof. By contradiction, assume that there exists $\varepsilon > 0$ such that

$$\|e^{t\mathcal{A}}u_0\|_{\mathcal{H}} \leq \frac{M}{t^{\frac{1}{l_k}+\varepsilon}}\|u_0\|_{D(\mathcal{A})}, \quad \forall t > 0. \tag{3.4.3}$$

It follows that

$$\|e^{t\mathcal{A}}u_0\|_{\mathcal{H}} \leq \frac{M}{t^{\frac{1}{l_k}+\varepsilon}}\|\mathcal{A}u_0\|_{\mathcal{H}}, \quad \forall t > 0. \tag{3.4.4}$$

Since $0 \in \rho(\mathcal{A})$ and $\mathcal{A}$ is onto over $\mathcal{H}$, then by taking $\mathcal{A}u_0 = f$, we get

$$\|e^{t\mathcal{A}}\mathcal{A}^{-1}f\|_{\mathcal{H}} \leq \frac{M}{t^{\frac{1}{l_k}+\varepsilon}}\|f\|_{\mathcal{H}}, \quad \forall f \in \mathcal{H}. \tag{3.4.5}$$

This implies that

$$\|e^{t\mathcal{A}}\mathcal{A}^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{M}{t^{\frac{1}{l_k}+\varepsilon}}, \quad \forall t > 0. \tag{3.4.6}$$

Using Proposition 3.1 in [8], we conclude that inequality (3.4.6) is equivalent to

$$\|e^{t\mathcal{A}}\mathcal{A}^{-\ell}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{M}{t}, \quad \forall t > 0 \tag{3.4.7}$$

where $\ell = \dfrac{l_k}{1+\varepsilon l_k}$. By applying Theorem 2.4 in [9] (see also [6]), we deduce, from (3.4.5), that

$$\|\mathrm{R}(\mathrm{is}, \mathcal{A})\|_{\mathcal{L}(\mathcal{H})} = O\left(|s|^{\ell}\right), \text{ as } s \longrightarrow \infty. \tag{3.4.8}$$

Now, set $\delta = \dfrac{1+2\varepsilon l_k}{2(1+\varepsilon l_k)}$. Assume that $l_k = \alpha_{\hat{k}}$, where $0 \leq \hat{k} \leq k_0$. Then, consider the sequences $\beta_n \subset \mathbb{R}$ and $(u_n) \subset D(\mathcal{A})$ as follows

$$\beta_n = \Im \lambda_{\hat{k},n},$$

$$u_n = e_{\hat{k},n}.$$

A direct calculation gives

$$\lim_{n\to+\infty} \beta_n^{l_k-\delta l_k} \|(i\beta_n - \mathcal{A})u_n\| = \lim_{n\to+\infty} \beta_n^{l_k-\delta l_k} |\Re \lambda_{\hat{k},n}| = \lim_{n\to+\infty} \mu_{\hat{k},n}^{-\delta l_k} = 0.$$

Consequently, there exists no constant $C > 0$ such that

$$\|\mathrm{R}(\mathrm{is}, \mathcal{A})\|_{\mathcal{L}(\mathcal{H})} < C|s|^{\frac{\ell}{2}}, \text{ as } s \longrightarrow \infty. \tag{3.4.9}$$

This contradicts the asymptotic equation (3.4.8) and the proof is thus complete.

□

Remark 3.4.2. *Theorem 3.4.1 gives a condition on the eigenvalues of the operator $\mathcal{A}$ that imply the optimality of a given polynomial energy decay estimate. In [24], Loreti and Rao proved that if* (3.4.1) *is verified and if the system of root vectors* $(e_{k,n})_{1\leq k\leq k_0, n\geq 1}$ *forms a Riesz basis in $\mathcal{H}$, then the polynomial decay estimate* (3.4.2) *is true and it is optimal.*

3.4.2 Lack of uniform stability result in the case $a = 1$ and $b = k\pi$.

In this subsection, we prove that the coupling parameter condition $b \neq k\pi$, $k \in \mathbb{Z}$ is necessary to the uniform stability of system (3.1.11)-(3.1.14). To be more precise, in the case $a = 1$ and if the coupling parameter b is on the form $b = k\pi$, $k \in \mathbb{Z}$, we show that there exists a branch of eigenvalues of $\mathcal{A}$ that is close to the imaginary axis.

Theorem 3.4.3. *Assume that $a = 1$, b satisfies (SC2) and there exists $k \in \mathbb{Z}$, such that $b = k\pi$. Then, the eigenvalues λ_m of the operator $\mathcal{A}$ satisfy the following asymptotic expansion*

$$\lambda_m = i\left(\mu_m + \frac{b^2}{8\mu_m} + \frac{7b^4}{128\mu_m^3}\right) - \frac{b^6}{256\mu_m^4} + O\left(\frac{1}{\mu_m^5}\right), \quad m \longrightarrow +\infty \tag{3.4.10}$$

where $\mu_m = m\pi$ if $\cos b = 1$ and $\mu_m = (2m+1)\frac{\pi}{2}$ if $\cos b = -1$.
In particular, the C_0 semigroup $e^{t\mathcal{A}}$ is not uniformly stable in the energy space $\mathcal{H}$.

Proof. **Step 1.** We start by looking for the eigenvalues of the operator $\mathcal{A}$. Let λ be an eigenvalue of $\mathcal{A}$ and $U = (u, v, y, z) \in D(\mathcal{A})$ its corresponding eigenvector. Then we have the following system

$$v = \lambda u, \tag{3.4.11}$$

$$u_{xx} - bz = \lambda v, \tag{3.4.12}$$

$$z = \lambda y, \tag{3.4.13}$$

$$y_{xx} + bv = \lambda z, \tag{3.4.14}$$

$$u(0) = u(1) = y(1) = y_x(0) - z(0) = 0. \tag{3.4.15}$$

Inserting equations (3.4.11) and (3.4.13) into (3.4.12) and (3.4.14) respectively, we get

$$u_{xx} - b\lambda y - \lambda^2 u = 0, \tag{3.4.16}$$

$$y_{xx} + b\lambda u - \lambda^2 y = 0, \tag{3.4.17}$$

$$u(0) = u(1) = y(1) = y_x(0) - \lambda y(0) = 0. \tag{3.4.18}$$

Combining (3.4.16), (3.4.17) and (3.4.18) we obtain

$$\begin{aligned} u_{xxxx} - 2\lambda^2 u_{xx} + (b^2\lambda^2 + \lambda^4)u &= 0, && (3.4.19)\\ u(0) = u(1) = u_{xx}(1) = u_{xxx}(0) - \lambda u_{xx}(0) - \lambda^2 u_x(0) &= 0. && (3.4.20) \end{aligned}$$

The general solution of (3.4.19)-(3.4.20) is given by $u(x) = \sum_{j=1}^{4} c_j e^{r_j x}$, where

$$c_j \in \mathbb{C},\ \ r_1 = \sqrt{\lambda^2 - i\lambda b},\ \ r_2 = -r_1,\ \ r_3 = \sqrt{\lambda^2 + i\lambda b} \text{ and } r_4 = -r_3.$$

The boundary conditions (3.4.20) can be written as

$$M(\lambda)\cdot C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ e^{r_1} & e^{r_2} & e^{r_3} & e^{r_4} \\ r_1^2 e^{r_1} & r_2^2 e^{r_2} & r_3^2 e^{r_3} & r_4^2 e^{r_4} \\ r_1 a_1 & r_2 a_2 & r_3 a_3 & r_4 a_4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0,$$

where

$$a_j = r_j^2 - \lambda r_j - \lambda^2, \qquad j = 1, ..4.$$

System (3.4.19)-(3.4.20) has no trivial solution if and only if the determinant of the matrix $M(\lambda)$ vanishes, equivalently $f(\lambda) = \frac{e^{2\lambda}}{8b^2\lambda^3}|M(\lambda)| = 0$.

Step. 2 Since the real part of λ is bounded (see Remark 3.4.4 below), then by using Maple software, we have

$$f(\lambda) = f_0(\lambda) + \frac{f_1(\lambda)}{\lambda} + \frac{f_2(\lambda)}{\lambda^2} + \frac{f_3(\lambda)}{\lambda^3} + \frac{f_4(\lambda)}{\lambda^4} + \frac{O(1)}{\lambda^5} \qquad (3.4.21)$$

where

$$\begin{aligned} f_0(\lambda) &= (\cos b - e^{2\lambda})e^{2\lambda}, && (3.4.22)\\ f_1(\lambda) &= -\frac{b^2}{4}e^{4\lambda}, && (3.4.23)\\ f_2(\lambda) &= \frac{b^2}{16}(1 - e^{4\lambda}) - \frac{b^4}{32}e^{4\lambda}, && (3.4.24)\\ f_3(\lambda) &= -\frac{b^4}{64} - \frac{b^6}{384}e^{4\lambda} + \frac{b^4}{16}(\cos b + e^{2\lambda})e^{2\lambda}, && (3.4.25)\\ f_4(\lambda) &= \frac{b^6}{512} + \frac{5b^4}{256}(e^{4\lambda} - 1) - \frac{b^8}{6144}e^{4\lambda} - \frac{b^6}{128}(\cos b)e^{2\lambda}. && (3.4.26) \end{aligned}$$

Furthermore, the roots of equation $f_0(\lambda) = 0$ are given by

$$\lambda_m^0 = i\mu_m, \qquad m \in \mathbb{Z} \tag{3.4.27}$$

where $\mu_m = m\pi$ if $\cos b = 1$ and $\mu_m = (2m+1)\frac{\pi}{2}$ if $\cos b = -1$. Since the real part of λ is bounded (see Remark (3.4.4)), then with the help of Rouché's theorem, and for λ large enough, we show that the roots of f are close to those of f_0 in other words there exists a sequence λ_m of roots of f such that

$$\lambda_m = i\mu_m + o(1) \qquad \text{as} \quad m \to +\infty. \tag{3.4.28}$$

This implies that the C_0-semigroup of contraction $e^{t\mathcal{A}}$ is not uniformly stable in the energy space $\mathcal{H}$. On the other hand, in order to obtain the optimal energy decay rate we will apply Theorem 3.4.1 by finding the real part of the eigenvalues λ_m.

Step 3. From Step 2, we can write

$$\lambda_m = i\mu_m + \epsilon_m, \ \text{ where } \ \epsilon_m = o(1). \tag{3.4.29}$$

Consequently, it follows from (3.4.22) and (3.4.23) that

$$f_0(\lambda_m) = -2\epsilon_m + O(\epsilon_m^2), \tag{3.4.30}$$

$$\frac{f_1(\lambda_m)}{\lambda_m} = i\frac{b^2}{4\mu_m} + i\frac{b^2}{\mu_m}\epsilon_m + O(\epsilon_m^2) + O\left(\frac{1}{\mu_m^2}\right). \tag{3.4.31}$$

Then, due to (3.4.21), (3.4.30), (3.4.31) and the fact that $f(\lambda_m) = 0$, we conclude

$$\epsilon_m = i\frac{b^2}{8\mu_m} + O\left(\frac{1}{\mu_m^2}\right). \tag{3.4.32}$$

Step 4. Due to (3.4.29) and (3.4.32), we have that

$$\lambda_m = i\mu_m + i\frac{b^2}{8\mu_m} + \hat{\epsilon}_m, \ \text{ where } \ \hat{\epsilon}_m = o(1). \tag{3.4.33}$$

Then, it follows from (3.4.22)-(3.4.26) that

$$
\begin{aligned}
f_0(\lambda_m) =& -2\hat{\epsilon}_m + i\frac{7b^6}{384\mu_m^3} + \frac{3b^4}{32\mu_m^2} - i\frac{b^2}{4\mu_m} \\
& + O(\hat{\epsilon}_m^2) + O\left(\frac{\hat{\epsilon}_m}{\mu_m}\right) + O\left(\frac{1}{\mu_m^4}\right),
\end{aligned}
\tag{3.4.34}
$$

$$
\frac{f_1(\lambda_m)}{\lambda_m} = -i\frac{b^4+b^6}{32\mu_m^3} - \frac{b^4}{8\mu_m^2} + i\frac{b^2}{4\mu_m} + O(\hat{\epsilon}_m^2) + O\left(\frac{\hat{\epsilon}_m}{\mu_m}\right) + O\left(\frac{1}{\mu_m^4}\right), \tag{3.4.35}
$$

$$
\frac{f_2(\lambda_m)}{\lambda_m^2} = i\frac{b^6+2b^4}{64\mu_m^3} + \frac{b^4}{32\mu_m^2} + O\left(\frac{\hat{\epsilon}_m}{\mu_m^2}\right) + O\left(\frac{1}{\mu_m^4}\right), \tag{3.4.36}
$$

$$
\frac{f_3(\lambda_m)}{\lambda_m^3} = i\frac{42b^4-b^6}{384\mu_m^3} + O\left(\frac{\hat{\epsilon}_m}{\mu_m^3}\right) + O\left(\frac{1}{\mu_m^4}\right), \tag{3.4.37}
$$

$$
\frac{f_4(\lambda_m)}{\lambda_m^4} = O\left(\frac{1}{\mu_m^4}\right). \tag{3.4.38}
$$

Inserting equations (3.4.34)-(3.4.38) into (3.4.21) and using the fact that $f(\lambda_m) = 0$, we get

$$
\hat{\epsilon}_m = i\frac{7b^4}{128\mu_m^3} + O(\frac{1}{\mu_m^4}). \tag{3.4.39}
$$

Step 5. Due to (3.4.39) and (3.4.33), we have that

$$
\lambda_m = i\left(\mu_m + \frac{b^2}{8\mu_m} + \frac{7b^4}{128\mu_m^3}\right) + \tilde{\epsilon}_m. \tag{3.4.40}
$$

Then, it follows from (3.4.22)-(3.4.26) that

$$
\begin{aligned}
f_0(\lambda_m) =& -2\tilde{\epsilon}_m + \frac{168b^6-5b^8}{2048\mu_m^4} + i\frac{7b^6-42b^4}{384\mu_m^3} + \frac{3b^4}{32\mu_m^2} - i\frac{b^2}{4\mu_m} \\
& + O(\tilde{\epsilon}_m^2) + O\left(\frac{\tilde{\epsilon}_m}{\mu_m}\right) + O\left(\frac{1}{\mu_m^5}\right),
\end{aligned}
\tag{3.4.41}
$$

$$
\begin{aligned}
\frac{f_1(\lambda_m)}{\lambda_m} =& \frac{2b^8-15b^6}{384\mu_m^4} - i\frac{b^4+b^6}{32\mu_m^3} - \frac{b^4}{8\mu_m^2} \\
& + i\frac{b^2}{4\mu_m} + O\left(\frac{\tilde{\epsilon}_m}{\mu_m}\right) + O\left(\frac{1}{\mu_m^5}\right),
\end{aligned}
\tag{3.4.42}
$$

$$
\frac{f_2(\lambda_m)}{\lambda_m^2} = -\frac{4b^6+b^8}{256\mu_m^4} + i\frac{2b^4+b^6}{64\mu_m^3} + \frac{b^4}{32\mu_m^2} + O\left(\frac{\tilde{\epsilon}_m}{\mu_m^2}\right) + O\left(\frac{1}{\mu_m^5}\right), \tag{3.4.43}
$$

$$
\frac{f_3(\lambda_m)}{\lambda_m^3} = \frac{b^8-36b^6}{768\mu_m^4} + i\frac{42b^4-b^6}{383\mu_m^3} + O\left(\frac{\tilde{\epsilon}_m}{\mu_m^3}\right) + O\left(\frac{1}{\mu_m^5}\right), \tag{3.4.44}
$$

$$
\frac{f_4(\lambda_m)}{\lambda_m^4} = \frac{72b^6-b^8}{6144\mu_m^4} + O\left(\frac{\tilde{\epsilon}_m}{\mu_m^4}\right) + O\left(\frac{1}{\mu_m^5}\right). \tag{3.4.45}
$$

Hence, inserting (3.4.41)-(3.4.45) into (3.4.21) and using the fact that $f(\lambda_m) = 0$, we get

$$\tilde{\epsilon}_m = -\frac{b^6}{256\mu_m^4} + O(\frac{1}{\mu_m^5}). \tag{3.4.46}$$

Finally, by (3.4.46) and (3.4.40), we deduce the desired asymptotic behavior equation (3.4.10). The proof is thus complete. □

Remark 3.4.4. *There exists a positive constant* C *such that for every eigenvalue* $\lambda = \alpha + i\beta$ *of the operator* $\mathcal{A}$*, we have* $0 \leq -\alpha \leq C$*. In fact, let* λ *be an eigenvalue of* $\mathcal{A}$ *and* $U = (u, \lambda u, y, \lambda y)$ *be an associated eigenvector such that* $\|U\| = 1$*. Multiplying equations (3.4.16) and (3.4.17) by* $\overline{u}$ *and* $\overline{y}$ *respectively and integrating by parts, then adding resulting equations, we get*

$$1 = -\alpha|y(0)|^2 + 2b\beta \mathrm{Im}\left(\int_0^1 y\overline{u}dx\right).$$

By taking $\beta = 0$ *and using trace theorem, we deduce the boundedness of* α*.*

Remark 3.4.5. *In the case* $b = \pi$*, we have* $-\frac{b^6}{256} \approx -3.75543$*. Then, the following table confirms the asymptotic behavior* (3.4.10)*:*

m	20	40	60	80	100	120
$\mu_m^4 \Re\lambda_k$	-3.75654	-3.75567	-3.75554	-3.75552	-3.75549	-3.75547

In the case $b = 2\pi$*, we have* $-\frac{b^6}{256} \approx -240.347$*. Then, the following table confirms the asymptotic behavior* (3.4.10)*:*

m	40	80	120	160	200	240
$\mu_m^4 \Re\lambda_k$	-240.422	-240.366	-240.318	-240.340	-240.345	-240.346

Remark 3.4.6. *In this remark we give a graphical interpretation of the lake of the exponential stability of system (3.1.11)-(3.1.14).*

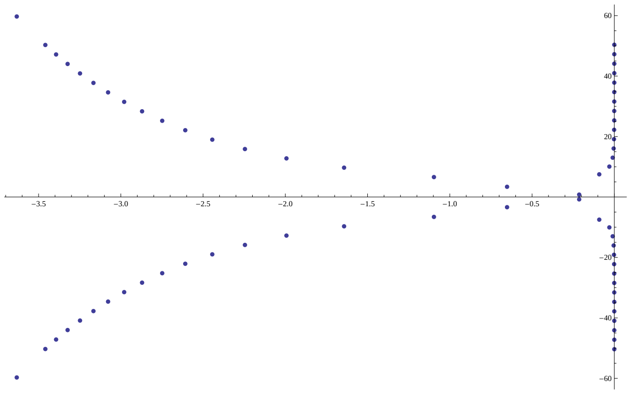

Figure 3.2: Eigenvalues in the case $a = 1$ and $b = 2\pi$

3.4.3 Optimal Polynomial stability in the case $a = 1$ and $b = k\pi$.

From previous theorem, if $a = 1$, b satisfies (SC2) and if there exists $k \in \mathbb{Z}$ such that $b = k\pi$, then system (3.1.11)-(3.1.14) is not uniformly stable, so it is natural to hope for a polynomial stability. To this end, using a frequency domain approach combining with a multiplier method, we establish the following result:

Theorem 3.4.7. *(Optimal polynomial energy decay rate) Assume that $a = 1$, b satisfies* (SC2) *and there exists $k \in \mathbb{Z}$, such that $b = k\pi$. Then there exists a constant $C > 0$ such that for every initial data $U_0 = (u_0, u_1, y_0, y_1) \in D(\mathcal{A})$, the energy of system (3.1.11)-(3.1.14) verify the following estimation:*

$$E(t) \leq C \frac{1}{\sqrt{t}} \|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0. \tag{3.4.47}$$

Moreover, the energy decay rate obtained in (3.4.47) *is optimal.*

Proof. Following Borichev and Tomilov [9], (see also [20], [6]), a C_0 semigroup of contractions $e^{t\mathcal{A}}$ on a Hilbert space $\mathcal{H}$ verify (3.4.47) if conditions (H1) and

$$\sup_{\lambda \in \mathbb{R}} \frac{1}{|\lambda|^4} \|(i\lambda I - \mathcal{A})^{-1}\| < +\infty \quad \text{(H3)}$$

are satisfied.

Condition (H1) was already proved, we will prove (H3) using an argument of contradiction. Suppose that (H3) is false, then there exist two sequences $(\lambda_n) \subset \mathbb{R}$ and $(U^n = (u^n, v^n, y^n, z^n)) \subset D(\mathcal{A})$, verifying the following conditions

$$|\lambda_n| \longrightarrow +\infty, \quad \|U^n\| = \|(u^n, v^n, y^n, z^n)\|_{\mathcal{H}} = 1, \tag{3.4.48}$$

$$\lambda_n^4 (i\lambda_n I - \mathcal{A})U^n = (f_1^n, f_2^n, g_1^n, g_2^n) \longrightarrow 0 \quad \text{in} \quad \mathcal{H}. \tag{3.4.49}$$

Since the sequence U_n is uniformly bounded in $\mathcal{H}$, then using equation (3.4.49), we get

$$|z^n(0)|^2 = Re((i\lambda_n I - \mathcal{A})U^n, U^n)_{\mathcal{H}} = \frac{o(1)}{\lambda^4}.$$

It follows that

$$|y_x^n(0)| = \frac{o(1)}{\lambda^2}. \tag{3.4.50}$$

Detailing equation (3.4.49), we get

$$\begin{aligned} i\lambda_n u^n - v^n &= \frac{f_1^n}{\lambda_n^4} \longrightarrow 0 \quad \text{in} \quad H_0^1, && (3.4.51) \\ i\lambda_n v^n - u_{xx}^n + bz^n &= \frac{g_1^n}{\lambda_n^4} \longrightarrow 0 \quad \text{in} \quad L^2, && (3.4.52) \\ i\lambda_n y^n - z^n &= \frac{f_2^n}{\lambda_n^4} \longrightarrow 0 \quad \text{in} \quad H_R^1, && (3.4.53) \\ i\lambda_n z^n - y_{xx}^n - bv^n &= \frac{g_2^n}{\lambda_n^4} \longrightarrow 0 \quad \text{in} \quad L^2. && (3.4.54) \end{aligned}$$

Using equation (3.4.53), we deduce

$$|y^n(0)| = \frac{o(1)}{\lambda^3}. \tag{3.4.55}$$

Since v^n and z^n are uniformly bounded in $L^2(0,1)$, then from equations (3.4.51) and (3.4.53), we deduce

$$\|u^n\| = \frac{O(1)}{\lambda} \text{ and } \|y^n\| = \frac{O(1)}{\lambda}. \tag{3.4.56}$$

Now, eliminate v^n and z^n in (3.4.52) and (3.4.54) by (3.4.51) and (3.4.53), we obtain the reduced system, where here and below for simplicity, we drop the index n

$$\lambda^2 u + u_{xx} - i\lambda b y = -\frac{g_1 + i\lambda f_1 + b f_2}{\lambda^4}, \tag{3.4.57}$$

$$\lambda^2 y + y_{xx} + i\lambda b u = -\frac{g_2 + i\lambda f_2 - b f_1}{\lambda^4}. \tag{3.4.58}$$

Using the same technique used in section 3.3, let $Y = (u, u_x, y, y_x)^T$, then system (3.4.57)-(3.4.58), could be written as

$$Y_x = BY + \frac{G}{\lambda^4} + \frac{F}{\lambda^3}, \tag{3.4.59}$$

where B, G and F are given by (3.3.20). Using Ordinary Differential Equation Theory, the solution of system (3.4.59) is given by

$$Y(x) = e^{Bx} Y_0 + \int_0^x e^{B(x-z)} \left(\frac{G(z)}{\lambda^4} + \frac{F(z)}{\lambda^3} \right) dz. \tag{3.4.60}$$

Under the same notations used in section 3.3 and performing advanced calculation for the exponential of the matrix B, we obtain the following

$$e^B = \begin{pmatrix} A_1 & 0 & iA_3 & 0 \\ \frac{b}{2}(\frac{b}{4}A_1 - A_4) & A_1 & \frac{ib}{2}(A_2 + \frac{b}{4}A_3) & iA_3 \\ -iA_3 & 0 & A_1 & 0 \\ \frac{-ib}{2}(A_2 + \frac{b}{4}A_3) & -iA_3 & \frac{b}{2}(\frac{b}{4}A_1 - A_4) & A_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\lambda A_2 & 0 & i\lambda A_4 & 0 \\ 0 & 0 & 0 & 0 \\ -i\lambda A_4 & 0 & -\lambda A_2 & 0 \end{pmatrix} + (o(1))$$

where $(o(1)) = (o_{ij}) \in \mathfrak{M}_{4\times4}(\mathbb{C})$ such that $o_{ij} = o(1)$. In particular, we have

$$o_{12} = \frac{1}{\lambda}A_2 - \frac{b}{2\lambda^2}(A_4 + \frac{b}{4}A_1) + \frac{b^2}{\lambda^3}[(\frac{-b^2}{128} + \frac{3}{8})A_2 - \frac{b}{8}A_3] + \frac{O(1)}{\lambda^4}, \tag{3.4.61}$$

$$o_{14} = \frac{-i}{\lambda}A_4 - \frac{ib}{2\lambda^2}(\frac{b}{4}A_3 - A_2) + \frac{O(1)}{\lambda^4}, \tag{3.4.62}$$

$$o_{32} = \frac{i}{\lambda}A_4 + \frac{ib}{2\lambda^2}(\frac{b}{4}A_3 - A_2) + \frac{ib}{8\lambda^3}[bA_1 + (\frac{-b^2}{16} + 3)A_4] + \frac{O(1)}{\lambda^4}, \tag{3.4.63}$$

$$o_{34} = \frac{1}{\lambda}A_2 - \frac{b}{2\lambda^2}(A_4 + \frac{b}{4}A_1) + \frac{O(1)}{\lambda^4}. \tag{3.4.64}$$

Since $G_1 = G_3 = 0$, $F_1 = F_3 = 0$, then using (3.4.61)-(3.4.64) and the fact that A_j, $j = 1, 2, 3, 4$, are uniformly bounded, we get

$$\int_0^x e^{B(x-z)} \left(\frac{G(z)}{\lambda^4} + \frac{F(z)}{\lambda^3}\right) dz = \left(\frac{o(1)}{\lambda^4}, \frac{o(1)}{\lambda^3}, \frac{o(1)}{\lambda^4}, \frac{o(1)}{\lambda^3}\right)^T. \tag{3.4.65}$$

On the other hand, using equations (3.4.50) and (3.4.55), we get

$$Y_0 = \left(0, u_x(0), \frac{o(1)}{\lambda^3}, \frac{o(1)}{\lambda^2}\right)^T. \tag{3.4.66}$$

Substituting equations (3.4.66) and (3.4.65) in (3.4.60) and take $x = 1$, we obtain

$$Y(1) = e^B Y_0 + \left(\frac{o(1)}{\lambda^4}, \frac{o(1)}{\lambda^3}, \frac{o(1)}{\lambda^4}, \frac{o(1)}{\lambda^3}\right)^T. \tag{3.4.67}$$

Combining (3.4.67) with (3.4.66), we get

$$\begin{aligned} 0 = \frac{1}{\lambda}\sin\lambda\cos\frac{b}{2}u_x(0) &- \frac{b}{2\lambda^2}\left(\sin\frac{b}{2} + \frac{b}{4}\cos\frac{b}{2}\right)\cos\lambda u_x(0) \\ &+ \frac{b^2}{\lambda^3}\left[(\frac{-b^2}{128} + \frac{3}{8})\cos\frac{b}{2} - \frac{b}{8}\sin\frac{b}{2}\right]\sin\lambda u_x(0) \\ &+ \frac{o(1)}{\lambda^3} \end{aligned} \tag{3.4.68}$$

and

$$\begin{aligned} 0 = \frac{1}{\lambda}\cos\lambda\sin\frac{b}{2}u_x(0) &+ \frac{b}{2\lambda^2}\left(\frac{b}{4}\sin\frac{b}{2} - \cos\frac{b}{2}\right)\sin\lambda u_x(0) \\ &+ \frac{b}{8\lambda^3}\left[b\cos\frac{b}{2} + (\frac{-b^2}{16} + 3)\sin\frac{b}{2}\right]\cos\lambda u_x(0) \\ &+ \frac{o(1)}{\lambda^3}. \end{aligned} \tag{3.4.69}$$

We distinguish two cases. Firstly, we consider the case $b = (2k+1)\pi$, $k \in \mathbb{Z}$. Then, multiplying equation (3.4.68) by λ^3, we get

$$\lambda \cos\lambda \sin\frac{b}{2} u_x(0) + \frac{b^2}{4} \sin\lambda \sin\frac{b}{2} u_x(0) = o(1). \tag{3.4.70}$$

It follows that

$$\cos\lambda \sin\frac{b}{2} u_x(0) = o(1). \tag{3.4.71}$$

Now, multiplying equation (3.4.69) by λ^2, we get

$$\lambda \cos\lambda \sin\frac{b}{2} u_x(0) + \frac{b^2}{8} \sin\lambda \sin\frac{b}{2} u_x(0) = o(1). \tag{3.4.72}$$

Combining equations (3.4.70) and (3.4.72), we get

$$\sin\lambda \sin\frac{b}{2} u_x(0) = o(1). \tag{3.4.73}$$

Adding the squares of equations (3.4.73) and (3.4.71), and using the fact that $\sin^2\frac{b}{2} = 1$, we deduce

$$u_x(0) = o(1). \tag{3.4.74}$$

Repeating Step 2 of Theorem3.3.1, we deduce that $\|U\|_{\mathcal{H}} = o(1)$, which contradicts (3.4.48) and the polynomial energy decay rate (3.4.47) is established. Secondly, we consider the case when $b = 2k\pi$, $k \in \mathbb{Z}$. Similarly, multiplying equation (3.4.68) by λ^2 and (3.4.69) by λ^3, we deduce that $u_x(0) = o(1)$. Consequently, $\|U\|_{\mathcal{H}} = o(1)$, which contradicts (3.4.48) and the polynomial energy decay rate (3.4.47) is established.

Finally, applying Theorem 3.4.1 by using the asymptotic expansion equation (3.4.10) (with $l_k = 4$), we deduce that the energy decay rate (3.4.47) is optimal in the sense that for any $\varepsilon > 0$, we cannot expect the energy decay rate $\frac{1}{t^{1/2+\varepsilon}}$. The proof is complete. $\square$

3.5 Polynomial Stability in the general case, $a \neq 1$

In this section, we study the asymptotic behavior of solutions of system (3.1.11)-(3.1.14) in the general case when $a \neq 1$. Our main result is the following theorem

Theorem 3.5.1. *(Polynomial energy decay rate) Assume that $a \neq 1$ and b satisfies condition (SC1). If $a \in \mathbb{Q}$ and b small enough* **or** *$\sqrt{a} \in \mathbb{Q}$. Then there exists a constant $C > 0$ such that for every initial data $U_0 = (u_0, u_1, y_0, y_1) \in D(\mathcal{A})$, the energy of system (3.1.11)-(3.1.14) verify the following estimate:*

$$E(t) \leq C \frac{1}{\sqrt{t}} \|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0. \tag{3.5.1}$$

Proof. By similar approach used in Theorem 3.4.7, we will check conditions (H1) and (H3). Condition (H1) was already proved in the general case in Theorem 3.2.3, we will prove (H3) using an argument of contradiction.
Suppose that (H3) is false, then there exist a sequence $(\lambda_n) \subset \mathbb{R}$ and a sequence $(U^n = (u^n, v^n, y^n, z^n)) \subset D(\mathcal{A})$, verifying the following conditions

$$|\lambda_n| \longrightarrow +\infty, \quad \|U^n\| = \|(u^n, v^n, y^n, z^n)\|_{\mathcal{H}} = 1, \tag{3.5.2}$$

$$\lambda_n^4 (i\lambda_n I - \mathcal{A}) U^n = (f_1^n, f_2^n, g_1^n, g_2^n) \longrightarrow 0 \quad \text{in} \quad \mathcal{H}. \tag{3.5.3}$$

Detailing equation (3.5.3), we get

$$i\lambda_n u^n - v^n = \frac{f_1^n}{\lambda_n^4} \longrightarrow 0 \quad \text{in} \quad H_0^1, \tag{3.5.4}$$

$$i\lambda_n v^n - u^n_{xx} + b z^n = \frac{g_1^n}{\lambda_n^4} \longrightarrow 0 \quad \text{in} \quad L^2, \tag{3.5.5}$$

$$i\lambda_n y^n - z^n = \frac{f_2^n}{\lambda_n^4} \longrightarrow 0 \quad \text{in} \quad H_R^1, \tag{3.5.6}$$

$$i\lambda_n z^n - a y^n_{xx} - b v^n = \frac{g_2^n}{\lambda_n^4} \longrightarrow 0 \quad \text{in} \quad L^2. \tag{3.5.7}$$

Multiply in $\mathcal{H}$ equation (3.5.3) by the uniformly bounded sequence $U^n = (u^n, v^n, y^n, z^n)$, we get

$$|z^n(0)|^2 = Re((i\lambda_n I - \mathcal{A})U^n, U^n)_{\mathcal{H}} = \frac{o(1)}{\lambda^4}.$$

It follows that

$$|y_x^n(0)| = \frac{o(1)}{\lambda^2}. \tag{3.5.8}$$

Using equation (3.5.6), we deduce

$$|y^n(0)| = \frac{o(1)}{\lambda^3}. \tag{3.5.9}$$

Since v^n and z^n are uniformly bounded in $L^2(0,1)$, then equations (3.5.4) and (3.5.6) give

$$\|u^n\| = \frac{O(1)}{\lambda} \quad \text{and} \quad \|y^n\| = \frac{O(1)}{\lambda}. \tag{3.5.10}$$

For simplicity, we drop the index n. Eliminating v and z in (3.5.5) and (3.5.7) by (3.5.4)and (3.5.6), we obtain the following reduced system

$$\lambda^2 u + u_{xx} - i\lambda b y = -\frac{g_1 + i\lambda f_1 + b f_2}{\lambda^4}, \tag{3.5.11}$$

$$\lambda^2 y + a y_{xx} + i\lambda b u = -\frac{g_2 + i\lambda f_2 - b f_1}{\lambda^4}, \tag{3.5.12}$$

Let $Y = (u, u_x, y, y_x)^T$, then system (3.5.11)-(3.5.12) could be written as

$$Y_x = BU + F \tag{3.5.13}$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\lambda^2 & 0 & i\lambda b & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{-i\lambda b}{a} & 0 & \dfrac{-\lambda^2}{a} & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ -\dfrac{b g_1 + i\lambda f_1 + f_2}{\lambda^4} \\ 0 \\ -\dfrac{g_2 + i\lambda g_1 - b f_1}{\lambda^4} \end{pmatrix} \quad \text{and} \quad Y_0 = \begin{pmatrix} 0 \\ u_x(0) \\ \dfrac{o(1)}{\lambda^3} \\ \dfrac{o(1)}{\lambda^2} \end{pmatrix}.$$

The solution of (3.5.13) at 1 is given by

$$Y(1) = e^B Y_0 + \int_0^1 e^{B(1-z)} F(z)\,dz, \tag{3.5.14}$$

where $e^B = (e_{ij})$ is the exponential of the matrix B. Using Maple software and the fact that the real part of λ is bounded a performing advanced calculation give

$$e^B = \begin{pmatrix} \cos(\lambda) & 0 & 0 & 0 \\ -\lambda\sin(\lambda) - \frac{b^2}{2(a-1)}\cos(\lambda) & \cos(\lambda) & \frac{ib}{(a-1)}\left[a\sin\lambda - \sqrt{a}\sin\left(\frac{\lambda}{\sqrt{a}}\right)\right] & 0 \\ 0 & 0 & \cos(\frac{\lambda}{\sqrt{a}}) & 0 \\ \frac{-ib}{a(a-1)}\left[a\sin\lambda - \sqrt{a}\sin\left(\frac{\lambda}{\sqrt{a}}\right)\right] & 0 & -\frac{\lambda}{\sqrt{a}}\sin\left(\frac{\lambda}{\sqrt{a}}\right) + \frac{b^2}{2(a-1)}\cos\left(\frac{\lambda}{\sqrt{a}}\right) & \cos\left(\frac{\lambda}{\sqrt{a}}\right) \end{pmatrix} + \left(\frac{O(1)}{\lambda}\right)$$

where $\left(\frac{O(1)}{\lambda}\right) = (o_{ij}) \in \mathfrak{M}_{4\times4}(\mathbb{C})$ such that $o_{ij} = \frac{O(1)}{\lambda}$. In particular, we have

$$o_{12} = \frac{\sin(\lambda)}{\lambda} + \frac{b^2}{2(a-1)\lambda^2}\cos(\lambda) + \frac{O(1)}{\lambda^3}, \tag{3.5.15}$$

$$o_{13} = \frac{iab}{(a-1)\lambda}\left(\cos\left(\frac{\lambda}{\sqrt{a}}\right) - \cos(\lambda)\right) + \frac{O(1)}{\lambda^2}, \tag{3.5.16}$$

$$o_{14} = \frac{iab}{(a-1)\lambda^2}\left[\sin(\lambda) + \sqrt{a}\sin\left(\frac{\lambda}{\sqrt{a}}\right)\right] + \frac{O(1)}{\lambda^3}, \tag{3.5.17}$$

$$\begin{aligned} o_{32} =& \frac{-ib}{(a-1)\lambda^2}\left[\sin(\lambda) + \sqrt{a}\sin\left(\frac{\lambda}{\sqrt{a}}\right)\right] \\ &- \frac{ib^3}{2(a-1)^2\lambda^3}\left[\cos(\lambda) - a\cos\left(\frac{\lambda}{\sqrt{a}}\right)\right] + \frac{O(1)}{\lambda^4}, \end{aligned} \tag{3.5.18}$$

$$o_{34} = \frac{\sqrt{a}}{\lambda}\sin\left(\frac{\lambda}{\sqrt{a}}\right) - \frac{ab^2}{2(a-1)\lambda^2}\cos\left(\frac{\lambda}{\sqrt{a}}\right) + \frac{O(1)}{\lambda^3}. \tag{3.5.19}$$

Since $F_1 = F_3 = 0$, then using e^B and the fact that functions f_j, g_j converge to zero in L^2, we get

$$\int_0^1 e^{B(1-z)}F(z)dz = \left(\frac{o(1)}{\lambda^4}, \frac{o(1)}{\lambda^3}, \frac{o(1)}{\lambda^4}, \frac{o(1)}{\lambda^3}\right)^T. \tag{3.5.20}$$

On the other hand, using equations (3.5.8) and (3.5.9)e^B, we get

$$Y_0 = \left(0, u_x(0), \frac{o(1)}{\lambda^3}, \frac{o(1)}{\lambda^2}\right)^T. \tag{3.5.21}$$

Our aim is to show that $u_x(0) = o(1)$, suppose that $u_x(0) = 1$. Then inserting e^B, (3.5.15)-(3.5.21) in (3.5.14) and use the fact that $u(1) = y(1) = 0$, we get

$$0 = \sin(\lambda) + \frac{b^2}{2(a-1)\lambda}\cos(\lambda) + \frac{O(1)}{\lambda^2} + \frac{o(1)}{\lambda^3}, \tag{3.5.22}$$

$$\begin{aligned} 0 =& \frac{ib}{a-1}\left(\sin(\lambda) + \sqrt{a}\sin\left(\frac{\lambda}{\sqrt{a}}\right)\right) \\ &- \frac{ib^3}{2(a-1)^2\lambda}\left(\cos(\lambda) - a\cos\left(\frac{\lambda}{\sqrt{a}}\right)\right) + \frac{O(1)}{\lambda^2} + \frac{o(1)}{\lambda}. \end{aligned} \tag{3.5.23}$$

Combining equation (3.5.22) and (3.5.23), we get

$$0 = \sin\left(\lambda + \frac{b^2}{2(a-1)\lambda}\right) + \frac{O(1)}{\lambda^2} + \frac{o(1)}{\lambda^3}, \tag{3.5.24}$$

$$0 = \sin\left(\frac{\lambda}{\sqrt{a}} - \frac{\sqrt{a}b^2}{2(a-1)\lambda}\right) + \frac{O(1)}{\lambda^2} + \frac{o(1)}{\lambda}. \tag{3.5.25}$$

It follows that there exist $n, m \in \mathbb{Z}$ such that

$$\lambda = n\pi - \frac{b^2}{2(a-1)\lambda} + \frac{O(1)}{\lambda^2} + \frac{o(1)}{\lambda^3}, \tag{3.5.26}$$

$$\frac{\lambda}{\sqrt{a}} = m\pi + \frac{\sqrt{a}b^2}{2(a-1)\lambda} + \frac{O(1)}{\lambda^2} + \frac{o(1)}{\lambda}. \tag{3.5.27}$$

Using the fact the λ is big enough, $\lambda \sim \pi n \sim \pi\sqrt{a}m$, then by taking the squares of equations (3.5.26) and (3.5.27) respectively, we get

$$\lambda^2 = n^2\pi^2 - \frac{b^2}{a-1} + \frac{O(1)}{\lambda}, \tag{3.5.28}$$

and

$$\lambda^2 = am^2\pi^2 + \frac{b^2 a}{a-1} + \frac{O(1)}{\lambda}. \tag{3.5.29}$$

Comparing equations (3.5.28) and (3.5.29), we get

$$\pi^2 n^2 - a\pi^2 m^2 = b^2\left(\frac{a+1}{a-1}\right) + \frac{O(1)}{\lambda}. \tag{3.5.30}$$

Case 1. Assume $a = \dfrac{p_0}{q_0}$ and $a \neq \dfrac{p^2}{q^2}$, for all , $p, q \in \mathbb{Z}$, then we have

$$|\frac{q_0 n^2 - p_0 m^2}{q_0}| \leq \frac{b^2}{\pi^2}(\frac{a+1}{a-1}) + \frac{O(1)}{\lambda}. \tag{3.5.31}$$

Since b is small enough, then we can assume that $b^2 \leq \frac{\pi^2(a-1)}{2(a+1)q_0}$. Consequently, using (3.5.31), we get the following contradiction

$$\frac{1}{2q_0} \leq \frac{1}{q_0} - \frac{b^2(1+a\sqrt{a})}{\pi(a-1)} \leq \frac{O(1)}{\lambda}. \tag{3.5.32}$$

Therefore, the system is polynomially stable for $a \in \mathbb{Q}$ and b small enough.

Case 2. Assume that there exist $p_0, q_0 \in \mathbb{Z}$ such that $a = \dfrac{p_0^2}{q_0^2}$. If $a = \dfrac{n^2}{m^2}$, then equation (3.5.30) gives the following contradiction

$$0 = b^2(\frac{a+1}{a-1}) + \frac{O(1)}{\lambda}.$$

Now, assume that

$$a = \frac{p_0^2}{q_0^2} \neq \frac{n^2}{m^2}.$$

Then equation (3.5.30) could be written as

$$(n - \frac{p_0}{q_0}m)(n + \frac{p_0}{q_0}m) = \frac{b^2}{\pi^2}(\frac{a+1}{a-1}) + \frac{O(1)}{\lambda}. \tag{3.5.33}$$

It follows that

$$\frac{nq_0 - mp_0}{q_0} = \left(\frac{b^2}{\pi^2}(\frac{a+1}{a-1})\right)\frac{q_0}{nq_0 + mp_0} + \frac{O(1)}{\lambda^2}. \tag{3.5.34}$$

This leads to the following contradiction.

$$\frac{1}{q_0} \leq \frac{O(1)}{\lambda}.$$

Therefore, the system is polynomially stable for $\sqrt{a} \in \mathbb{Q}$. □

Remark 3.5.2. *In this remark we give a numerical test of the polynomial stability of system (3.1.11)-(3.1.14) in the case $a \in \mathbb{Q}$ and $\sqrt{a} \in \mathbb{Q}$.*

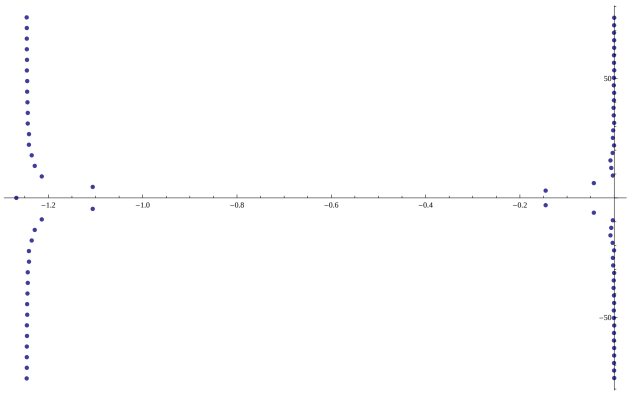

Figure 3.3: Eigenvalues in the case $a = 4$ and $b = 1$

Remark 3.5.3. *In this remark we give a numerical test of the polynomial stability of system (3.1.11)-(3.1.14) in the case $a \in \mathbb{Q}$ but $\sqrt{a} \notin \mathbb{Q}$.*

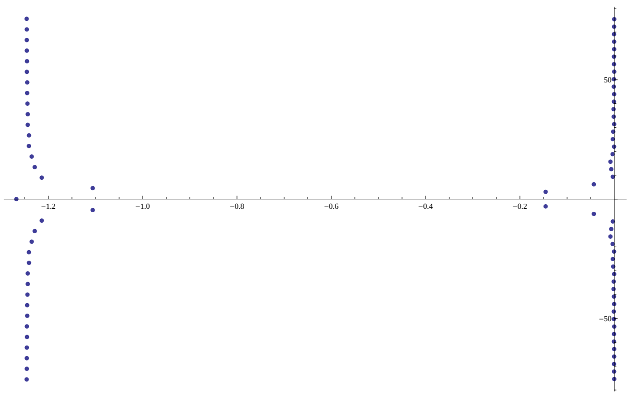

Figure 3.4: Eigenvalues in the case $a = 2$ and $b = 1$

Conclusion

We have studied the influence of the coefficients on the indirect boundary stabilization of a system of wave equations coupled via the velocity terms. If the wave speeds are equal ($a = 1$) and if the coupling parameter b is not on the form $k\pi$, $k \in \mathbb{Z}$ and it is outside a discrete set of exceptional values, using a frequency domain approach combining with a multiplier method, we have proved an uniform stability. Moreover, if the coupling parameter b is on the form $k\pi$, $k \in \mathbb{Z}$ a non uniform stability is proved and an optimal polynomial energy decay rate of type $\frac{1}{\sqrt{t}}$ is established. In the general case, when $a \neq 1$ a non uniform stability is expected but it is only checked numerically in some examples. Finally, if $\sqrt{a}$ is a rational number and if b is outside another discrete set of exceptional values, using a frequency domain approach, we proved a polynomial energy decay rate of type $\frac{1}{\sqrt{t}}$. We conjecturer that the remaining cases could be analyzed in the same way with a slower polynomial decay rate. In fact, we consider the following numerical test

Remark 3.5.4. *In this remark we give a numerical test of the polynomial stability of system (3.1.11)-(3.1.14) in the case* $a \notin \mathbb{Q}$ *and* $\sqrt{a} \notin \mathbb{Q}$.

Figure 3.5: Eigenvalues in the case $a = \sqrt{2}$ and $b = 1$

This will be investigated in the future.

Chapter 4

Indirect stability of a system of weakly coupled wave equations with local Kelvin-Voigt damping

4.1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary Γ. We consider the following system of coupled wave equations with a viscoelastic damping around the boundary Γ:

$$\begin{cases} \varrho_1(x)u_{tt} - \operatorname{div}(a_1(x)\nabla u + b(x)\nabla u_t) + \alpha y = 0, & \text{in} \quad \Omega \times \mathbb{R}^+, \\ \varrho_2(x)y_{tt} - \operatorname{div}(a_2(x)\nabla y) + \alpha u = 0, & \text{in} \quad \Omega \times \mathbb{R}^+, \\ u = y = 0, & \text{on} \quad \Gamma \times \mathbb{R}^+, \\ u(0) = u_0, \ y(0) = y_0 \ u_t(0) = u_1, \ y_t(0) = y_1, & \text{in} \quad \Omega, \end{cases} \tag{4.1.1}$$

where $\varrho_1(x) \geq \varrho_1 > 0$, $\varrho_2(x) \geq \varrho_2 > 0$, $a_1(x) \geq a_1^0 > 0$, $a_2(x) \geq a_2^0 > 0$, and $b(x) \geq 0$ for all $x \in \Omega$, the coupling parameter α is a real number.

Let $U = (u, u_t, y, y_t)$ be a regular solution of system (4.1.1). Then, the total natural energy of the system is given by:

$$\begin{aligned} E(t) = 1/2 \int_\Omega (\varrho_1(x)|u_t|^2 + a_1(x) \mid \nabla u \mid^2 + \varrho_2(x)|y_t|^2 \\ + a_2(x) \mid \nabla y \mid^2 + \alpha u y) dx. \end{aligned} \tag{4.1.2}$$

By a straightforward calculation we obtain that

$$E'(t) = -\int_\Omega b(x) \mid \nabla u_t \mid^2 dx \leq 0.$$

That is the system (4.1.1) is dissipative in the sense that its energy is decreasing with respect to the time t.

The local viscoelastic damping is a natural phenomena of bodies which have one part made of viscoelastic material, and the other is made of elastic material. There are a few number of publications concerning the wave equation with local viscoelastic damping. In [19], Liu and Rao studied the stability of a wave equation with local viscoelastic damping distributed around the boundary of the domain. They proved that the energy of the system goes exponentially to zero for all usual initial data. K. Liu and Z. Liu in [18], considered the longitudinal and transversal vibrations of the Euler-Bernoulli beam with Kelvin-Voigt damping distributed locally on any subinterval of the region occupied by the beam. They proved that the semigroup associated with the equation for the transversal motion of the beam is exponentially stable, although the semigroup associated with the equation for the longitudinal motion of the beam is not exponentially stable.

In this chapter, we consider a system of wave equations which are weakly coupled and partially damped by one locally distributed Kelvin-Voigt damping. The first equation is effectively damped, the second equation is indirectly damped through the coupling parameter. Firstly, using a unique continuation result based on a Carleman estimate, we show that the system is strongly stable for all usual initial data. Secondly, using a spectral approach, we show that the system is not uniformly exponentially stable. Then, it is natural to expect a polynomial energy decay rate. For this aim, using a frequency domain approach combined with piece wise multiplier method, we establish a polynomial energy decay rate.

Now we give a brief outline of the chapter. In section 2, using a unique continuation result and a general criteria of Arendt in [5], we show the strong stability of the system in the absence of the compactness of the resolvent. In addition, using a spectrum approach, we prove the non uniform stability of the system. In section 3, by a frequency domain approach combined with a piece wise multiplier method, we establish a polynomial energy decay rate as $1/\sqrt[4]{t}$ for smooth solutions.

4.2 Abstract setting and strong stability

Let $\alpha_0 = \min\left(\frac{a_1^0}{c_0^2}, \frac{a_2^0}{c_0^2}\right)$ where c_0^2 is the Poincaré constant. In what follows, we assume that α is a real number such that $|\alpha| < \alpha_0$. In this section, we suppose that coefficient functions ϱ_1, ϱ_2, a_1, b, $a_2 \in L^\infty(\Omega)$.

For any $\gamma > 0$, we define the γ-neighborhood O_γ of the boundary Γ as follows

$$O_\gamma := \{x \in \Omega : \inf_{y\in\Gamma} |x - y| \leq \gamma, \}, \tag{4.2.1}$$

and assume that there exist two constants b_0 and γ such that

$$b(x) \geq b_0 > 0, \qquad \forall x \in O_\gamma. \qquad \text{(SC)}$$

We start by formulate system (4.1.1) as an abstract Cauchy problem in an appropriate Hilbert space. First, define the energy space $\mathcal{H}$ by

$$\mathcal{H} = (H_0^1(\Omega) \times L^2(\Omega))^2 \tag{4.2.2}$$

endowed with the inner product:

$$(U, V)_{\mathcal{H}} = \int_\Omega (a_1(x)\nabla u\cdot\nabla\tilde{u}+a_2(x)\nabla y\cdot\nabla\tilde{y})dx+\int_\Omega(\varrho_1 v\tilde{v}+\varrho_2 z\tilde{z})dx+\int_\Omega \alpha(u\tilde{y}+y\tilde{u})dx$$

for all $U = (u, v, y, z)$, $V = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z}) \in \mathcal{H}$.

Next, define the unbounded linear operator $\mathcal{A}$ by :

$$D(\mathcal{A}) = \big\{(u, v, y, z) \in \mathcal{H} : \mathrm{div}(a_2(x)\nabla y), \mathrm{div}(a_1(x)\nabla u + b(x)\nabla v) \in L^2(\Omega), \text{ and } v, z \in H_0^1(\Omega)\big\}$$

and

$$\mathcal{A}U = (v, \frac{1}{\varrho_1}(\mathrm{div}(a_1(x))\nabla u + b(x)\nabla v) - \frac{\alpha}{\varrho_1}y, z, \frac{1}{\varrho_2}\mathrm{div}(a_2(x)\nabla y) - \frac{\alpha}{\varrho_2}u)$$

for all $U = (u, v, y, z) \in D(\mathcal{A})$. If $U = (u, u_t, y, y_t)$ is a regular solution of system (4.1.1), then we rewrite this system as the following evolutionary equation:

$$U_t = \mathcal{A}U, \quad U(0) = U_0 \in \mathcal{H}. \tag{4.2.3}$$

Proposition 4.2.1. *The unbounded linear operator $\mathcal{A}$ is dissipative and invertible in the energy space $\mathcal{H}$.*

Proof. First, let $U = (u, v, y, z) \in D(\mathcal{A})$. Then we have

$$\mathrm{Re}(AU, U)_{\mathcal{H}} = -\int_{\Omega} b(x) \mid \nabla v \mid^2 dx \leq 0. \tag{4.2.4}$$

Which implies that $\mathcal{A}$ is dissipative. Next, let $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}$, find $U = (u, v, y, z) \in D(\mathcal{A})$ solution of the equation

$$-AU = F. \tag{4.2.5}$$

Equivalently, we have the following system

$$-v = f_1, \tag{4.2.6}$$

$$-\mathrm{div}(a_1(x)\nabla u + b(x)\nabla v) + \alpha y = \varrho_1 f_2, \tag{4.2.7}$$

$$-z = f_3, \tag{4.2.8}$$

$$-\mathrm{div}(a_2(x)\nabla y) + \alpha u = \varrho_2 f_4. \tag{4.2.9}$$

Multiply equation (4.2.7) by $\varphi \in H_0^1(\Omega)$ and equation (4.2.9) by $\psi \in H_0^1(\Omega)$ respectively and integrate by parts over Ω, we get

$$\begin{aligned}&\int_{\Omega}(a_1(x)\nabla u \cdot \nabla\varphi + a_2(x)\nabla y \cdot \nabla\psi + \alpha y\varphi + \alpha u\psi)dx \\ &= \int_{\Omega}(\varrho_1(x) f_2\varphi + b(x)\nabla f_1 \cdot \nabla\varphi + \varrho_2(x) f_4\psi)dx.\end{aligned} \tag{4.2.10}$$

Using Lax-Milgramm theorem, the variational problem (4.2.10) admits a unique solution $(u, y) \in H_0^1(\Omega) \times H_0^1(\Omega)$. Taking $\varphi \in D(\Omega)$ and $\psi \equiv 0$, we deduce that $\mathrm{div}(a_1(x)\nabla u - b(x)\nabla f_1) \in L^2(\Omega)$ and similarly we deduce that $\mathrm{div}(a_2(x)\nabla y) \in L^2(\Omega)$. Finally, set $v = -f_1$ and $z = -f_3$, we deduce that $U = (u, v, y, z) \in D(\mathcal{A})$ is a solution of equation (4.2.5) and the proof is thus complete. □

Now, from the contraction principle, we easily get $R(\lambda I - \mathcal{A}) = \mathcal{H}$ for small $\lambda > 0$ (see [23]). This, together with the dissipativeness of $\mathcal{A}$, imply the density of $D(\mathcal{A})$ in $\mathcal{H}$ (see [27], Theorem 1.4.6). Then $\mathcal{A}$ generates a C_0-semigroup of contraction $e^{t\mathcal{A}}$ (see [27]) and we have the following result:

Theorem 4.2.2. *(Existence and uniqueness of the solution).*
(1) If $U_0 = (u_0, u_1, y_0, y_1) \in D(\mathcal{A})$, then problem (4.2.3) admits a strong unique solution $U = (u, v, y, z)$ such that :

$$U(t) \in C^1(\mathbb{R}_+, \mathcal{H}) \cap C^0(\mathbb{R}_+, D(\mathcal{A})).$$

(2) If $U_0 = (u_0, u_1, y_0, y_1) \in \mathcal{H}$, then problem (4.2.3) admits a unique week solution $U = (u, v, y, z)$ such that :

$$U(t) \in C^0(\mathbb{R}_+, \mathcal{H}).$$

Now, our objective is to study the asymptotic behavior of the solution of system (4.2.3).

4.2.1 Strong Stability with non compact resolvent

The goal of this section is to study the strong stability of system (4.2.3). We prove the following result:

Theorem 4.2.3. *Assume that (SC) holds and a_1, $a_2 \in C^{0,1}(\overline{\Omega})$. Then the C_0-semigoup $e^{t\mathcal{A}}$ is strongly stable on the energy space $\mathcal{H}$ in the sense that*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}} U_0\| = 0, \quad \forall U_0 \in \mathcal{H}.$$

Proof. The resolvent of $\mathcal{A}$ is not compact. Then classical methods such as Lasalle's invariance principle [31] or the spectrum decomposition theory of Benchimol [7] are not applicable in this case. We prove the strong stability with a more general criteria of Arendt in [5] following which a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in a Banach space is strongly stable, if $\mathcal{A}$ has no pure imaginary eigenvalues and if the set $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable. For simplicity we divide the proof into several steps.

Step 1. In this step we prove, by a contradiction argument, that $\mathrm{Ker}(i\lambda - \mathcal{A}) = \{0\}$ for all $\lambda \in \mathbb{R}$. From Proposition 2.1 we deduce that $0 \in \rho(\mathcal{A})$, then assume that $\lambda \neq 0$ and let $U = (u, v, y, z) \in D(\mathcal{A})$ such that

$$\mathcal{A}U = i\lambda U. \tag{4.2.11}$$

Using equation (4.2.4) we get

$$0 = \mathrm{Re}(i\lambda \|U\|^2) = \mathrm{Re}(\mathcal{A}U, U) = -\int_\Omega b(x)|\nabla v|^2 dx. \tag{4.2.12}$$

This together with the condition (SC) and Poincaré's inequality imply that

$$b(x)\nabla v \equiv 0 \text{ in } \Omega \text{ and } v \equiv 0 \text{ in } O_\gamma. \tag{4.2.13}$$

Inserting (4.2.13) into (4.2.11), we get

$$\begin{cases} v = i\lambda u, & \text{in } \Omega, \\ \mathrm{div}(a_1(x)\nabla u) - \alpha y = i\lambda \rho_1 v, & \text{in } \Omega, \\ z = i\lambda y, & \text{in } \Omega, \\ \mathrm{div}(a_2(x)\nabla y) - \alpha u = i\lambda \rho_2 z, & \text{in } \Omega. \end{cases} \tag{4.2.14}$$

The first equation in (4.2.14) gives that $u \equiv 0$ in O_γ. From the second and third equations in (4.2.14) we deduce respectively that $y \equiv 0$ and $z \equiv 0$ in O_γ. Then we have the following system

$$\begin{cases} L_1 u = \rho_1 \lambda^2 u + \mathrm{div}(a_1(x)\nabla u) - \alpha y = 0, & \text{in } \Omega, \\ L_2 y = \rho_2 \lambda^2 y + \mathrm{div}(a_2(x)\nabla y) - \alpha u = 0, & \text{in } \Omega, \\ u = y = 0, & \text{in } O_\gamma. \end{cases} \tag{4.2.15}$$

Using the fact that $a_1, a_2 \in C^{0,1}(\overline{\Omega})$, we deduce that the solution (u, y) of system (4.2.15) belongs to $H_0^2(\Omega) \times H_0^2(\Omega)$.

Step 2. Let $\varphi(x) = \dfrac{|x - x_0|^2}{2}$, for $x_0 \notin \overline{\Omega}$ and set $P_m(x, \xi) = |\xi|^2$ in Theorem 8.3.1 in [14] we deduce that there exist $C > 0$ and $\tau_0 >> 1$ such that for all $\tau > \tau_0$, we have

$$\tau^3 \int_\Omega e^{2\tau\varphi}|f|^2 dx + \tau \int_\Omega e^{2\tau\varphi}|\nabla f|^2 dx \leq C_1 \int_\Omega e^{2\tau\varphi}|\Delta f|^2 dx, \ \ \forall f \in C_0^\infty(\Omega). \tag{4.2.16}$$

By a density argument we extend equation (4.2.16) into the space of $H_0^2(\Omega)$. Then, take respectively $f = u$ and $f = y$ in equation (4.2.16) and combining the resulting equations, we get

$$\begin{aligned} &\tau^3 \int_\Omega e^{2\tau\varphi}(|u|^2 + |y|^2)dx + \tau \int_\Omega e^{2\tau\varphi}(|\nabla u|^2 + |\nabla y|^2)dx \\ &\leq C_2 \int_\Omega e^{2\tau\varphi}(|a_1(x)\Delta u|^2 + |a_2(x)\Delta y|^2)dx, \ \ \forall \tau > \tau_0. \end{aligned} \tag{4.2.17}$$

Combining the first and second equation of (4.2.15) with (4.2.17) and using the fact that $a_1, a_2 \in C^{0,1}(\overline{\Omega})$, we deduce that there exist positive constants C_3, C_4 and C_5 such that

$$\begin{aligned} (\tau^3 - C_3) \int_\Omega e^{2\tau\varphi}(|u|^2 + |y|^2)dx + (\tau - C_4) \int_\Omega e^{2\tau\varphi}(|\nabla u|^2 + |\nabla y|^2)dx \\ \leq C_5 \int_\Omega e^{2\tau\varphi}(|L_1 u|^2 + |L_2 y|^2)dx, \ \ \forall \tau > \tau_0. \end{aligned} \tag{4.2.18}$$

Choosing $\tau^3 - C_3 \geq \frac{1}{2}$ and $\tau - C_4 \geq \frac{1}{2}$, we deduce that $u \equiv 0$ and $y \equiv 0$ in Ω.

Step 3. The aim of this step is to prove that $\mathrm{R}(i\lambda I - \mathcal{A}) = \mathcal{H}$. Then, let $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}$ we solve the equation

$$i\lambda U - \mathcal{A}U = F. \tag{4.2.19}$$

This involves $v = i\lambda u - f_1$, $z = i\lambda y - f_3$ and the following system

$$\begin{cases} \lambda^2 u + \dfrac{1}{\rho_1}\mathrm{div}(\mathrm{a_1(x)\nabla u + i\lambda b(x)\nabla u}) - \dfrac{\alpha}{\rho_1}\mathrm{y} = \mathrm{f}, & \text{in } \Omega, \\ \\ \lambda^2 y + \dfrac{1}{\rho_2}\mathrm{div}(a_2(x)\nabla y) - \dfrac{\alpha}{\rho_2}u = g, & \text{in } \Omega \end{cases} \tag{4.2.20}$$

where $f = -f_2 - i\lambda f_1 + \frac{1}{\rho_1}\mathrm{div}(b(x)\nabla f_1) \in H^{-1}(\Omega)$ and $g = -f_4 - i\lambda f_3 \in L^2(\Omega)$. Now, define the linear operator $\mathsf{A} : H_0^1(\Omega) \times H_0^1(\Omega) \longrightarrow H^{-1}(\Omega) \times H^{-1}(\Omega)$ by

$$\mathsf{A}\begin{pmatrix} u \\ y \end{pmatrix} := \begin{pmatrix} -\dfrac{1}{\rho_1}\mathrm{div}(a_1(x)\nabla u + i\lambda b(x)\nabla u) + \dfrac{\alpha}{\rho_1} y \\ \\ -\dfrac{1}{\rho_2}\mathrm{div}(a_2(x)\nabla y) + \dfrac{\alpha}{\rho_2} u \end{pmatrix}. \tag{4.2.21}$$

It is easy to see that the operator A is an isomorphism from $H_0^1(\Omega) \times H_0^1(\Omega)$ onto $H^{-1}(\Omega) \times H^{-1}(\Omega)$. Let $\mathsf{U} = (u, y)^T$ and $\mathsf{F} = (f, g)^T$, then we transform system (4.2.20) into the following form

$$\mathsf{U} - \lambda^2 \mathsf{A}^{-1}\mathsf{U} = -\mathsf{A}^{-1}\mathsf{F} = \widetilde{\mathsf{F}}. \tag{4.2.22}$$

The operator $\mathsf{A}^{-1} : L^2 \times L^2 \to H^{-1} \times H^{-1} \to H_0^1 \times H_0^1 \to L^2 \times L^2$ is compact, then using Fredholm's alternative (see [10]), the equation (4.2.22) admits a unique solution $\mathsf{U} \in L^2 \times L^2$ if and and only if $I - \lambda^2 \mathsf{A}^{-1}$ is injective. Then, let $\mathsf{U} \in L^2 \times L^2$ such that

$$\mathsf{U} - \lambda^2 \mathsf{A}^{-1}\mathsf{U} = 0. \tag{4.2.23}$$

Equivalently, we have

$$\begin{cases} \lambda^2 u + \dfrac{1}{\rho_1}\mathrm{div}(a_1(x)\nabla u + i\lambda b(x)\nabla u) - \dfrac{\alpha}{\rho_1} y = 0, & \text{in } \Omega, \\ \\ \lambda^2 y + \dfrac{1}{\rho_2}\mathrm{div}(a_2(x)\nabla y) - \dfrac{\alpha}{\rho_2} u = 0, & \text{in } \Omega, \\ \\ u = y = 0, & \text{on } \Gamma. \end{cases} \tag{4.2.24}$$

Multiplying the first and the second equations of (4.2.24) by $\rho_1 \overline{u}$ and $\rho_2 \overline{y}$ and integrating by parts, we get

$$\begin{aligned} \lambda^2 \int_\Omega (\rho_1(x)|u|^2 + \rho_2(x)|y|^2)dx - \int_\Omega (a_1(x)|\nabla u|^2 + a_2(x)|\nabla y|^2 \\ + i\lambda b(x)|\nabla u|^2)dx - \alpha \int_\Omega (y\overline{u} + u\overline{y})dx = 0. \end{aligned} \tag{4.2.25}$$

Taking the imaginary part of equation (4.2.25) and using Poincaré's inequality, we deduce that $b(x)\nabla u \equiv 0$ in Ω and $u \equiv 0$ in O_γ. Using Step 1 and Step 2, we deduce that $u \equiv 0$ and $y \equiv 0$ in Ω and consequently equation (4.2.23) admits $U = 0$ as a unique solution. This implies that equation (4.2.22) admits a unique solution $\mathsf{U} = (u, y) \in H_0^1 \times H_0^1$ and $\operatorname{div}(a_2(x)\nabla y)$, $\operatorname{div}(a_1(x)\nabla u + i\lambda b(x)\nabla u - b(x)\nabla f_1) \in L^2$. Set $v = i\lambda u - f_1$ and $z = i\lambda y - f_3$, we deduce that $U = (u, v, y, z) \in D(\mathcal{A})$ is the unique solution of equation (4.2.19) and the proof si thus complete. □

Remark 4.2.4. *We mention [29] for a direct approach of the strong stability of Kirchhoff plates in the absence of compactness of the resolvent.*

4.2.2 Non uniform stability of the system

Theorem 4.2.5. *For any $\epsilon > 0$, we cannot expect the energy decay rate $\dfrac{1}{t^{1/2+\epsilon}}$ for all initial data $U_0 \in D(\mathcal{A})$. In particular, the C_0-semigroup $e^{t\mathcal{A}}$ is not uniformly exponentially stable in the energy space, and there exists $k_0 \in \mathbb{N}^*$ sufficiently large such that*

$$\sigma(\mathcal{A}) \supset \left\{ \lambda_k^1 = i\mu_k - \frac{\alpha^2}{2\mu_k^4} + o\left(\frac{1}{2\mu_k^5}\right) \right\}. \tag{4.2.26}$$

Proof. To prove the preceding theorem, we need the asymptotic behavior of the eigenvalues of the operator $\mathcal{A}$. First, let λ be an eigenvalue of $\mathcal{A}$ and $\Phi = (u, v, y, z)$ be an associated eigenfunction, then we have

$$\mathcal{A}\Phi = \lambda\Phi. \tag{4.2.27}$$

We consider the particular case $\rho_1 = \rho_2 = a = b = 1$. Equation (4.2.27) equivalent to $v = \lambda u$, $z = \lambda y$ and

$$\begin{cases} \lambda^2 u - \Delta u - \lambda \Delta u + \alpha y = 0, & \text{in} \quad \Omega, \\ \lambda^2 y - \Delta y + \alpha u = 0, & \text{in} \quad \Omega, \\ u = y = 0, & \text{on} \quad \Gamma. \end{cases} \tag{4.2.28}$$

By elimination of u, system (4.2.28) leads to

$$\begin{cases} \Delta^2 y - \lambda^2 \Big(\dfrac{2+\lambda}{1+\lambda} \Big) \Delta y + \dfrac{\lambda^4 - \alpha^2}{1+\lambda} y = 0, & \text{in} \quad \Omega, \\ y = \Delta y = 0, & \text{on} \quad \Gamma. \end{cases} \tag{4.2.29}$$

Now, let φ_k be an normalized eigenfunction of the following problem

$$\begin{cases} -\Delta \varphi_k = \mu_k^2 \varphi_k, & \text{in} \quad \Omega, \\ \varphi_k = 0, & \text{on} \quad \Gamma. \end{cases} \tag{4.2.30}$$

Then by taking $y = \varphi_k$ in (4.2.29), we deduce the following characteristic equation:

$$p(\lambda) = \lambda^4 + \mu_k^2 \lambda^3 + 2\mu_k^2 \lambda^2 + \mu_k^4 \lambda + \mu_k^4 - \alpha^2 = 0. \tag{4.2.31}$$

Let $\xi = \frac{\lambda}{\mu_k}$ and $\zeta_k = \frac{1}{\mu_k}$. From the second equation of system (4.2.28), we deduce that $\frac{\lambda}{\mu_k}$ is bounded. Then, multiplying equation (4.2.31) by $\frac{1}{\mu_k^5}$, we get

$$\xi^3 + \xi + \zeta_k + 2\xi^2 \zeta_k + \xi^4 \zeta_k - \alpha^2 \zeta_k^5 = 0. \tag{4.2.32}$$

Since $\frac{\lambda}{\mu_k}$ is bounded and $\zeta_k \to 0$, then thanks to Rouché's theorem, there exists k_0 large enough such that for all $|k| \geq k_0$ the large roots of the polynomial p are close to the roots of the polynomial $p_0(\xi) = \xi^3 + \xi$.

$$\xi_k^1 = i + \epsilon_k \ \text{ and } \ \lambda_k^1 = i\mu_k + \mu_k \epsilon_k, \quad \lim_{|k| \to \infty} \epsilon_k = 0. \tag{4.2.33}$$

Inserting equation (4.2.33) into equation (4.2.32), we get

$$\epsilon_k = o(\frac{1}{\mu_k}) \ \text{ and } \ \lambda_k^1 = i\mu_k + \widetilde{\epsilon_k}, \quad \lim_{|k| \to \infty} \widetilde{\epsilon_k} = 0. \tag{4.2.34}$$

Multiplying equation (4.2.31) by $\frac{1}{\mu_k^4}$ and inserting (4.2.34) in the resulting equation, we obtain

$$\widetilde{\epsilon_k} = o(\frac{1}{\mu_k^2}) \text{ and } \lambda_k^1 = i\mu_k + \frac{1}{\mu_k^2}\widehat{\epsilon_k}, \ \ \widehat{\epsilon_k} = o(1). \tag{4.2.35}$$

Again Multiplying equation (4.2.31) by $\frac{1}{\mu_k^2}$ and inserting (4.2.35) in the resulting equation, we obtain the desired result in (4.2.26) and the proof is thus complete. □

4.3 Polynomial Stability

The system (4.1.1) is not uniformly stable as we showed previously. In this section we prove that system (4.1.1) is polynomially stable in the energy space $\mathcal{H}$. Throughout this part, we assume that

$$a_1, a_2, \rho_1, \rho_2, b \in C^{1,1}(\bar{\Omega}). \tag{H1}$$

Also, we assume the following supplementary conditions.
There exist two functions $q, \hat{q} \in C^1(\Omega, \mathbb{R}^N)$ and $0 < \alpha < \beta < \gamma$, such that

$$\partial_j q_k = \partial_k q_j, \quad \mathrm{div}(\mathrm{a}_2\rho_2\mathrm{q}) \in \mathrm{C}^{0,1}(\Omega_\beta) \quad \text{and} \quad \mathrm{q} = 0 \ \text{ on } \ \mathrm{O}_\alpha, \tag{H2}$$

$$\partial_j \hat{q}_k = \partial_k \hat{q}_j, \quad \mathrm{div}(\mathrm{a}_1\rho_1\hat{\mathrm{q}}) \in \mathrm{C}^{0,1}(\Omega_\beta) \quad \text{and} \quad \hat{\mathrm{q}} = 0 \ \text{ on } \ \mathrm{O}_\alpha, \tag{H3}$$

There exists a constant $\sigma_1 > 0$, such that

$$2a_2\partial q_j + (q_k\partial_j a_2 + q_j\partial_k a_2) + \left[\frac{a_2}{\rho_2}(q\nabla\rho_2 - q\nabla a_2)\right] I \geq \sigma_1 I, \quad \forall x \in \Omega_\beta. \tag{H4}$$

There exists a constant $\sigma_2 > 0$, such that

$$2a_1\partial\hat{q} + (\hat{q}_k\partial_j a_1 + \hat{q}_j\partial_k a_1) + \left[\frac{a_1}{\rho_1}(\hat{q}\nabla\rho_1 - \hat{q}\nabla a_1)\right] I \geq \sigma_2 I, \quad \forall x \in \Omega_\beta. \tag{H5}$$

There exists a constant $M > 0$ such that for all $v \in H_0^1(\Omega)$, we have

$$|(\hat{q}\cdot\nabla v)\nabla b - (\hat{q}\cdot\nabla b)\nabla v| \leq M\sqrt{b}\ |\nabla v|, \quad \forall x \in \Omega_\beta. \tag{H6}$$

Theorem 4.3.1. *(Polynomial energy decay rate)*
Assume that conditions (SC), (H1)- (H6) *are satisfied. Then, there exists a constant* $C > 0$ *such that for every initial data* $U_0 = (u_0, u_1, y_0, y_1) \in D(\mathcal{A})$, *the energy of system (4.1.1) verify the following estimate:*

$$E(t) \leq C \frac{1}{\sqrt[4]{t}} \|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0. \tag{4.3.1}$$

Proof. Following Borichev and Tomilov [9], (see also [20], [6]), a C_0 semigroup of contractions $e^{t\mathcal{A}}$ on a Hilbert space $\mathcal{H}$ verify (4.3.1) if

$$i\mathbb{R} \subset \rho(\mathcal{A}) \qquad \text{(H1)}$$

and

$$\frac{1}{\lambda^8} \|(i\lambda - \mathcal{A})^{-1}\| < +\infty \qquad \text{(H2)}.$$

We know that condition (H1) is verified. Our goal now is to prove that condition (H2) is satisfied. To this aim, we proceed by a contradiction argument. Suppose that (H2) does not hold, then there exist a sequence $(\lambda_n) \subset \mathbb{R}$ and a sequence $(U_n) \subset D(\mathcal{A})$ such that

$$|\lambda_n| \longrightarrow +\infty, \qquad \|U_n\|_{\mathcal{H}} = \|(u_n, v_n, y_n, z_n)\|_{\mathcal{H}} = 1 \tag{4.3.2}$$

and

$$\lambda_n^8 (i\lambda_n - \mathcal{A}) U_n = (f_1^n, g_1^n, f_2^n, g_2^n) \longrightarrow 0 \quad \text{in } \mathcal{H} \tag{4.3.3}$$

are satisfied.

Step 1. *The dissipation.*

Multiply in $\mathcal{H}$ equation (4.3.3) by the uniformly bounded sequence $U^n = (u_n, v_n, y_n, z_n)$, we get

$$\int_\Omega b(x)|\nabla v_n| = -Re((i\lambda_n I - \mathcal{A})U^n, U^n)_{\mathcal{H}} = \frac{o(1)}{\lambda^8}.$$

It follows that

$$\|b(x)\nabla v_n\| = \frac{o(1)}{\lambda^4} \quad \text{in } \ L^2(\Omega), \tag{4.3.4}$$

Using (SC) and Poincaré inequality, we get

$$\|\nabla v_n\| = \frac{o(1)}{\lambda^4} \quad \text{in} \quad L^2(O_\gamma), \text{and} \quad \|v_n\| = \frac{o(1)}{\lambda^4} \quad \text{in} \quad L^2(O_\gamma). \tag{4.3.5}$$

In what follows, we drop the index n for simplicity.

Step 2. *First information on u and y.*

By detailing equation (4.3.3), we get the following system

$$i\lambda u - v = \frac{f_1}{\lambda^8} \longrightarrow 0 \quad \text{in} \quad H_0^1(\Omega), \tag{4.3.6}$$

$$i\lambda\rho_1 v - \mathrm{div}(\mathrm{a}_1\nabla \mathrm{u} + \mathrm{b}\nabla \mathrm{v}) + \alpha \mathrm{y} = \frac{\rho_1 \mathrm{g}_1}{\lambda^8} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\Omega), \tag{4.3.7}$$

$$i\lambda y - z = \frac{f_2}{\lambda^8} \longrightarrow 0 \quad \text{in} \quad H_0^1(\Omega), \tag{4.3.8}$$

$$i\lambda\rho_2 z - \mathrm{div}(\mathrm{a}_2\nabla \mathrm{y}) + \alpha \mathrm{u} = \frac{\rho_2 \mathrm{g}_2}{\lambda^8} \longrightarrow 0 \quad \text{in} \quad \mathrm{L}^2(\Omega). \tag{4.3.9}$$

From equations (4.3.2), (4.3.6), (4.3.8) and (4.3.5), we get

$$\|u\|_{L^2(\Omega)} = \frac{O(1)}{\lambda}, \quad \|y\|_{L^2(\Omega)} = \frac{O(1)}{\lambda} \quad \text{and} \quad \|u\|_{L^2(O_\gamma)} = \frac{o(1)}{\lambda^5}. \tag{4.3.10}$$

Step 3. Let $\varepsilon > 0$ such that $0 < \alpha < \beta < \gamma - \varepsilon < \gamma$. We define the cut-off function $\eta \in C_c^1(\overline{\Omega})$ by

$$\eta = \begin{cases} 1 & \text{on } O_{\gamma-\varepsilon}, \\ 0 & \text{on } \Omega_\gamma, \\ 0 \leq \eta \leq 1. \end{cases}$$

Multiply equation (4.3.7) by $\eta\overline{u}$ and integrate over Ω, we get

$$\begin{aligned} i\lambda \int_{O_\gamma} \rho_1 v\eta\overline{u}dx + \int_{O_\gamma} a_1\eta|\nabla u|^2 dx + \int_{O_\gamma} a_1\nabla u \cdot \nabla\eta\overline{u}dx \\ + \int_{O_\gamma} b\nabla v \cdot \nabla(\eta\overline{u})dx + \int_{O_\gamma} \alpha y\eta\overline{u} = \frac{o(1)}{\lambda^4}. \end{aligned} \tag{4.3.11}$$

Combining (4.3.5) and (4.3.10) with (4.3.11), we get

$$\|\nabla u\|_{L^2(O_{\gamma-\varepsilon})} = \frac{o(1)}{\lambda^2}. \tag{4.3.12}$$

Now, multiply (4.3.7) by $\lambda^2\eta\overline{y}$, we get

$$\begin{aligned} i\lambda^3\int_{O_\gamma}\rho_1 v\eta\overline{y}dx+\lambda^2\int_{O_\gamma}a_1\eta\nabla u\cdot\nabla\overline{y}dx+\lambda^2\int_{O_\gamma}a_1\nabla u\cdot\nabla\eta\overline{y}dx \\ +\lambda^2\int_{O_\gamma}b\nabla v\cdot\nabla(\eta\overline{y})dx+\int_{O_\gamma}\alpha\eta|\lambda y|^2=o(1). \end{aligned} \tag{4.3.13}$$

Using (4.3.5), (4.3.12), we get

$$\|\lambda y\|_{L^2(O_{\gamma-\varepsilon})}=o(1). \tag{4.3.14}$$

Inserting equation (4.3.8) into equation (4.3.9), we get

$$-\lambda^2\rho_2 y-\mathrm{div}(\mathrm{a_2}\nabla\mathrm{y})+\alpha\mathrm{u}=\frac{\rho_2\mathrm{g_2}+\mathrm{i}\lambda\mathrm{f_2}}{\lambda^8}. \tag{4.3.15}$$

Multiply equation (4.3.15) by $\eta\overline{y}$ and integrate over Ω, we get

$$-\int_{O_\gamma}\rho_2\eta|\lambda y|^2+\int_{O_\gamma}a_2\eta|\nabla y|^2+\alpha\int_{O_\gamma}u\eta\overline{y}=o(1). \tag{4.3.16}$$

Using (4.3.10) and (4.3.14), we get

$$\|\nabla y\|_{L^2(O_{\gamma-\varepsilon})}=o(1). \tag{4.3.17}$$

Step 4. Define

$$N=a_2\nabla y. \tag{4.3.18}$$

Eliminating z in (4.3.9) by (4.3.8) and multiplying the resulting equation by $q\cdot\overline{N}$, we get

$$-\lambda^2\int_\Omega\rho_2 yq\cdot\bar{N}dx-\int_\Omega\mathrm{div(N)q\cdot\bar{N}dx}+\alpha\int_\Omega\mathrm{uq\cdot\bar{N}}=\frac{\mathrm{o(1)}}{\lambda^8}, \tag{4.3.19}$$

(i) Using Green's formula, we get

$$-\lambda^2\Re\int_\Omega\rho_2 yq\cdot\bar{N}dx=-\lambda^2\Re\int_\Omega a_2\rho_2 yq\cdot\nabla\bar{y}dx=\frac{1}{2}\int_\Omega\mathrm{div}(\rho_2\mathrm{a_2q})|\lambda\mathrm{y}|^2\mathrm{dx}. \tag{4.3.20}$$

Now, let $h\in C^{0,1}(\overline{\Omega})$. Eliminate z in (4.3.9) by (4.3.8) and multiplying the resulting equation by $h\overline{y}$, we get

$$-\int_\Omega h|\lambda y|^2dx+\int_\Omega\frac{a_2}{\rho_2}h|\nabla y|^2dx+\int_\Omega a_2\overline{y}\nabla h\cdot\nabla ydx=o(1). \tag{4.3.21}$$

Take $h = \mathrm{div}(\rho_2 \mathrm{a}_2 \mathrm{q})$. Then, using (4.3.10), (4.3.20), (4.3.21) and the fact that ∇y is uniformly bounded in $L^2(\Omega)$, we deduce

$$-\lambda^2 \Re \int_\Omega \rho_2 y q \cdot \bar{N} dx = \int_\Omega \frac{a_2}{2\rho_2} \mathrm{div}(\rho_2 \mathrm{a}_2 \mathrm{q}) |\nabla \mathrm{y}|^2 \mathrm{dx} + \mathrm{o}(1). \tag{4.3.22}$$

(ii) We will estimate the second integral in

$$\begin{aligned} -\Re \int_\Omega \mathrm{div}(\mathrm{N}) \mathrm{q} \cdot \bar{\mathrm{N}} \mathrm{dx} &= -\Re \int_\Omega \partial_j N_j q_k \bar{N}_k dx \\ &= \Re \int_\Omega (N_j \partial_j q_k \bar{N}_k + N_j q_k \partial_j \bar{N}_k) dx \\ &= \Re \int_\Omega (N_j \partial_j q_k \bar{N}_k + N_j q_k \partial_k \bar{N}_j) dx \\ &+ \Re \int_\Omega N_j q_k (\partial_j \bar{N}_k - \partial_k \bar{N}_j)) dx. \end{aligned} \tag{4.3.23}$$

Using Green's formula, we get

$$\begin{aligned} -\Re \left\{ \int_\Omega (q \cdot \bar{N}) \mathrm{div}(\mathrm{N}) \mathrm{dx} \right\} &= \Re \left\{ \int_\Omega \left(N_j \partial_j q_k \bar{N}_k - \frac{1}{2} |N|^2 div(q) \right) dx \right. \\ &\left. + \int_\Omega N \cdot [(q \cdot \nabla \bar{y}) \nabla a_2 - (q \cdot \nabla a_2) \nabla \bar{y}] dx \right\}. \end{aligned} \tag{4.3.24}$$

(iii) Inserting equation (4.3.22), (4.3.24) in (4.3.19), we get

$$\begin{aligned} \int_\Omega \frac{a_2}{2\rho_2} \mathrm{div}(\rho_2 \mathrm{a}_2 \mathrm{q}) |\nabla \mathrm{y}|^2 \mathrm{dx} + \Re \left\{ \int_\Omega \left(\mathrm{N_j} \partial_\mathrm{j} \mathrm{q_k} \bar{\mathrm{N}}_\mathrm{k} - \frac{1}{2} |\mathrm{N}|^2 \mathrm{div}(\mathrm{q}) \right) \mathrm{dx} \right. \\ \left. + \int_\Omega N \cdot [(q \cdot \nabla \bar{y}) \nabla a_2 - (q \cdot \nabla a_2) \nabla \bar{y}] dx \right\} = o(1). \end{aligned} \tag{4.3.25}$$

It follows that

$$\begin{aligned} \int_\Omega \frac{a_2}{2\rho_2} \mathrm{div}(\rho_2 \mathrm{a}_2 \mathrm{q}) |\nabla \mathrm{y}|^2 \mathrm{dx} + \Re \left\{ \int_\Omega \mathrm{a}_2^2 \left(\partial_\mathrm{j} \mathrm{y} \partial_\mathrm{j} \mathrm{q_k} \partial_\mathrm{k} \mathrm{y} - \frac{1}{2} \mathrm{div}(\mathrm{q}) |\nabla \mathrm{y}|^2 \right) \mathrm{dx} \right. \\ \left. + \int_\Omega a_2 \nabla y \cdot [(q \cdot \nabla \bar{y}) \nabla a_2 - (q \cdot \nabla a_2) \nabla \bar{y}] dx \right\} = o(1). \end{aligned} \tag{4.3.26}$$

A direct calculation, gives

$$\begin{aligned} \int_\Omega \frac{a_2}{2\rho_2} \mathrm{div}(\rho_2 \mathrm{a}_2 \mathrm{q}) |\nabla \mathrm{y}|^2 \mathrm{dx} + \Re \left\{ \int_\Omega \mathrm{a}_2^2 \left(\partial_\mathrm{j} \mathrm{y} \partial_\mathrm{j} \mathrm{q_k} \partial_\mathrm{k} \bar{\mathrm{y}} - \frac{1}{2} \mathrm{div}(\mathrm{q}) |\nabla \mathrm{y}|^2 \right) \mathrm{dx} \right. \\ \left. + \int_\Omega [\frac{a_2}{2} (q_k \partial_j a_2 + q_j \partial_k a_2) \partial_j y \partial_k \bar{y} - a_2 q \cdot \nabla a_2 |\nabla y|^2] dx \right\} = o(1). \end{aligned} \tag{4.3.27}$$

This implies that

$$\begin{aligned}&\int_\Omega \frac{a_2}{2}\left[(2a_2\partial_j q_k + q_k\partial_j a_2 + q_j\partial_k a_2)\partial_j y\partial_k\overline{y}\right]dx\\&+\int_\Omega \frac{a_2}{2}(\frac{a_2}{\rho_2}q\cdot\nabla\rho_2 - q\cdot\nabla a_2)|\nabla y|^2dx = o(1).\end{aligned} \tag{4.3.28}$$

Using condition (H4) in equation (4.3.28), we deduce that

$$\int_{\Omega_\beta}|\nabla y|^2dx = o(1). \tag{4.3.29}$$

It follows from (4.3.17), (4.3.21) and use the fact that ∇y is uniformly bounded, we deduce

$$\int_\Omega|\nabla y|^2dx = o(1) \text{ and } \int_\Omega|\lambda y|^2dx = o(1). \tag{4.3.30}$$

Step 5. We define the following multiplier

$$M = a_1(x)\nabla u + b(x)\nabla v. \tag{4.3.31}$$

Multiplying (4.3.7) by the uniformly bounded sequence $\hat{q}\cdot\overline{M}$, we get

$$i\lambda\int_\Omega\rho_1 v\hat{q}\cdot\bar{M}dx - \int_\Omega \mathrm{div(M)\hat{q}\cdot\bar{M}dx} + \alpha\int_\Omega \mathrm{y\hat{q}\cdot\bar{M}} = \frac{\mathrm{o(1)}}{\lambda^8}, \tag{4.3.32}$$

(i) Using (4.3.4), (4.3.6) and the fact that ∇u is uniformly bounded in $L^2(\Omega)$, we get

$$\begin{aligned}\Re\int_\Omega i\lambda\rho_1 v\hat{q}\cdot\bar{M}dx =& -\lambda^2\Re\int_\Omega a_1\rho_1 u\hat{q}\cdot\nabla\bar{u}dx + o(1)\\=&\frac{1}{2}\int_\Omega \mathrm{div(\rho_1 a_1\hat{q})|\lambda u|^2dx}.\end{aligned} \tag{4.3.33}$$

Now, let $\hat{h}\in C^{0,1}(\overline{\Omega})$. Multiplying equation (4.3.6) by $i\lambda\hat{h}\overline{u}$ and (4.3.7) by $\hat{h}\overline{u}$ respectively. Then using (4.3.4), (4.3.10) and the fact that ∇u is uniformly bounded, we obtain

$$-\int_\Omega\hat{h}|\lambda u|^2dx - i\lambda\int_\Omega\hat{h}v\bar{u}dx = \frac{o(1)}{\lambda^8}, \tag{4.3.34}$$

$$i\lambda\int_\Omega\hat{h}v\bar{u}dx + \int_\Omega\frac{a_1}{\rho_1}\hat{h}|\nabla u|^2dx = o(1). \tag{4.3.35}$$

Take $\hat{h} = \mathrm{div}(\rho_1 \mathrm{a}_1 \hat{\mathrm{q}})$. Then, combining (4.3.34) with (4.3.35), we deduce

$$\Re \int_\Omega i\lambda \rho_1 v \hat{q} \cdot \bar{M} dx = \int_\Omega \frac{a_1}{2\rho_1} \mathrm{div}(\rho_1 \mathrm{a}_1 \hat{\mathrm{q}}) |\nabla \mathrm{u}|^2 \mathrm{dx} + \mathrm{o}(1). \tag{4.3.36}$$

(ii) We will estimate the second integral in (4.3.32). Using Green's formula, we get

$$\begin{aligned} -\Re \int_\Omega \mathrm{div}(\mathrm{M}) \hat{\mathrm{q}} \cdot \bar{\mathrm{M}} \mathrm{dx} &= -\Re \int_\Omega \partial_j M_j \hat{q}_k \bar{M}_k dx \\ &= \Re \int_\Omega (M_j \partial_j \hat{q}_k \bar{M}_k + M_j \hat{q}_k \partial_j \bar{M}_k) dx \\ &= \Re \int_\Omega (M_j \partial_j \hat{q}_k \bar{M}_k + M_j \hat{q}_k \partial_k \bar{M}_j) dx \\ &+ \Re \int_\Omega M_j \hat{q}_k (\partial_j \bar{M}_k - \partial_k \bar{M}_j)) dx + o(1). \end{aligned} \tag{4.3.37}$$

Using Green's formula in equation (4.3.37), we get

$$-\Re \int_\Omega \mathrm{div}(\mathrm{M})(\hat{\mathrm{q}} \cdot \bar{\mathrm{M}}) \mathrm{dx} = \Re \int_\Omega (M_j \partial_j \hat{q}_k \bar{M}_k - \frac{1}{2} |M|^2 \mathrm{div}(\hat{\mathrm{q}})) \mathrm{dx}$$

$$+\Re \int_\Omega M \cdot [(\hat{q} \cdot \nabla \bar{u}) \nabla a_1 - (\hat{q} \cdot \nabla a_1) \nabla \bar{u}] dx \tag{4.3.38}$$

$$+\Re \int_\Omega M \cdot [(\hat{q} \cdot \nabla \bar{v}) \nabla b - (\hat{q} \cdot \nabla b) \nabla \bar{v}] dx.$$

(iii) Using condition (H6), we deduce that the last integral in (4.3.38) is $o(1)$. Then, inserting equation (4.3.36), (4.3.38) in (4.3.32) and use the fact that $\|y\| = o(1)$, we get

$$\begin{aligned} \int_\Omega \frac{a_1}{2\rho_1} \mathrm{div}(\rho_1 \mathrm{a}_1 \hat{\mathrm{q}}) |\nabla \mathrm{u}|^2 \mathrm{dx} + \Re \Bigg\{ \int_\Omega \left(\mathrm{M_j} \partial_\mathrm{j} \hat{\mathrm{q}}_\mathrm{k} \bar{\mathrm{M}}_\mathrm{k} - \frac{1}{2} |\mathrm{M}|^2 \mathrm{div}(\hat{\mathrm{q}}) \right) \mathrm{dx} \\ + \int_\Omega M \cdot [(\hat{q} \cdot \nabla \bar{u}) \nabla a_1 - (\hat{q} \cdot \nabla a_1) \nabla \bar{u}] dx \Bigg\} = o(1). \end{aligned} \tag{4.3.39}$$

It follows that

$$\begin{aligned} \int_\Omega \frac{a_1}{2\rho_1} \mathrm{div}(\rho_1 \mathrm{a}_1 \hat{\mathrm{q}}) |\nabla \mathrm{u}|^2 \mathrm{dx} + \Re \Bigg\{ \int_\Omega \mathrm{a}_1^2 \left(\partial_\mathrm{j} \mathrm{u} \partial_\mathrm{j} \hat{\mathrm{q}}_\mathrm{k} \partial_\mathrm{k} \mathrm{u} - \frac{1}{2} \mathrm{div}(\hat{\mathrm{q}}) |\nabla \mathrm{u}|^2 \right) \mathrm{dx} \\ + \int_\Omega a_1 \nabla u \cdot [(\hat{q} \cdot \nabla \bar{u}) \nabla a_1 - (\hat{q} \cdot \nabla a_1) \nabla \bar{u}] dx \Bigg\} = o(1). \end{aligned} \tag{4.3.40}$$

A direct calculation, gives

$$\begin{aligned}&\int_\Omega \frac{a_1}{2\rho_1}\mathrm{div}(\rho_1 \mathrm{a}_1\hat{\mathrm{q}})|\nabla \mathrm{u}|^2 \mathrm{dx} + \Re\Big\{\int_\Omega \mathrm{a}_1^2\left(\partial_{\mathrm{j}}\mathrm{u}\partial_{\mathrm{j}}\hat{\mathrm{q}}_{\mathrm{k}}\partial_{\mathrm{k}}\overline{\mathrm{u}} - \frac{1}{2}\mathrm{div}(\hat{\mathrm{q}})|\nabla \mathrm{u}|^2\right)\mathrm{dx}\\ &\quad + \int_\Omega [\frac{a_1}{2}(\hat{q}_k\partial_j a_1 + \hat{q}_j\partial_k a_1)\partial_j u\partial_k\overline{u} - a_1\hat{q}\cdot\nabla a_1|\nabla u|^2]dx\Big\} = o(1).\end{aligned} \tag{4.3.41}$$

This implies that

$$\begin{aligned}&\int_\Omega \frac{a_1}{2}\left[(2a_1\partial_j\hat{q}_k + \hat{q}_k\partial_j a_1 + \hat{q}_j\partial_k a_1)\partial_j u\partial_k\overline{u}\right]dx\\ &+\int_\Omega \frac{a_1}{2}(\frac{a_1}{\rho_1}\hat{q}\cdot\nabla\rho_1 - \hat{q}\cdot\nabla a_1)|\nabla u|^2 dx = o(1).\end{aligned} \tag{4.3.42}$$

Using condition (H5) in equation (4.3.42), we deduce that

$$\int_{\Omega_\beta}|\nabla u|^2 dx = o(1). \tag{4.3.43}$$

It follows from (4.3.12), (4.3.34) and (4.3.35) that

$$\int_\Omega |\nabla u|^2 dx = o(1) \text{ and } \int_\Omega |\lambda u|^2 dx = o(1). \tag{4.3.44}$$

Step 6. (Conclusion)

Combining (4.3.30) with (4.3.44), we deduce that $\|U\| = o(1)$. The proof is thus complete.

□

Conclusion

We have studied the stabilization of a system of wave equations coupled via the zero order terms with one locally distributed Kelvin-Voigt damping. First we prove that the system is not exponentially stable. Next, using a frequency domain approach combining with a multiplier method, we have established a polynomial stability. The damping is localized around the boundary of the domain, then the polynomial stability of the system when the damping is localized on any subdomain of the global domain remains an open problem.

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