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Étude de la stabilité de quelques systèmes d'équations des ondes couplées sur des domaines bornés et non bornés

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Contents

Introduction

La thèse porte essentiellement sur la stabilisation indirecte d'un système de deux équations des ondes couplées et sur la stabilisation frontière de poutre de Rayleigh.

Dans le cas de la stabilisation d'un système d'équations d'onde couplées, le contrôle est introduit dans le système directement sur le bord du domaine d'une seule équation dans le cas d'un domaine borné ou dans le cas d'un domaine non borné à l'interieur d'une seule équation. La nature du système ainsi couplé dépend du couplage des équations et de la nature arithmétique des vitesses de propagation. Ceci donne divers résultats de stabilisation polynômiale ainsi que de non stabilité.

Dans le cas de la stabilisation de la poutre de Rayleigh, l'équation est considerée avec un seul contrôle force agissant sur le bord du domaine. D'abord, moyennant le développement asymptotique des valeurs propres et des vecteurs propres du système non contrôlé, un résultat d'oservabilité ainsi qu'un résultat de bornetude de la fonction de transfert correspendant sont obtenus. Alors, un taux de décroissance polynomial de l'énergie du système est établi. Ensuite, moyennant une étude spectrale combinée avec une méthode fréquentielle, l'optimalité du taux obtenu est assurée.

Chapitre 1 . Stabilité frontière indirecte du système de Timoshenko

Dans ce Chapitre nous étudions la stabilité frontière indirecte du système du Timoshenko suivant :

$$\begin{cases} u_{tt} - (u_x + y)_x = 0 & \text{dans} \quad (0,1) \times (0, \infty), \\ y_{tt} - a y_{xx} + b(u_x + y) = 0 & \text{dans} \quad (0,1) \times (0, \infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{dans} \quad (0,1), \\ y(x,0) = y_0(x), y_t(x,0) = y_1(x) & \text{dans} \quad (0,1), \\ u(0,t) = u(1,t) = y(1,t) = 0 & \text{dans} \quad (0, \infty) \end{cases} \tag{0.0.1}$$

avec la loi de dissipation

$$a y_x(0,t) = b y_t(0,t) \qquad \text{dans} \quad (0, \infty), \tag{0.0.2}$$

où a, b sont des constantes positives. Les fonctions u et y désignent, respectivement, le déplacement transversal de la poutre et la rotation angulaire de coupe transversale.

La notion de mécanisme de dissipation indirecte a été introduite par Russell [45], et depuis lors, elle a retenu l'attention de plusieurs auteurs, par exemple, citons les articles de Alabau [2, 3] pour des études générales sur des systèmes hyperboliques avec stabilisation au bord indirecte. Signalons que notre système n'entre pas dans le cadre de ces papiers. Rappelons maintenant quelques résultats concernant la stabilisation frontière du système de Timoshenko. Dans [25], Kim et Renardy ont établi la stabilisation exponentielle du système moyennant deux contrôles frontières. Dans [10], Ammar-Khodja et al. ont étudié la stabilisation frontière du système de Timoshenko non uniforme en appliquant deux contrôles frontières sur l'équation de rotation angulaire. Sous la condition d'égalité de vitesse de propagation ($a = 1$), ils ont établi un taux de décroissance exponentiel de l'énergie dans un sous-espace orthogonal à un sous espace de dimension finie mais non précisé. En revanche, quand les vitesses sont différentes ils ont prouvé la stabilisation forte et non exponentielle mais aucun taux de décroissance de l'énergie n'est discuté.

Dans ce Chapitre, nous étudions la stabilisation du système (0.0.1)-(0.0.2) avec un seul contrôle frontière agissant sur l'équation de rotation angulaire. D'abord, nous montrons que le système est fortement stable si et seulement si le terme de couplage b n'appartient pas à un ensemble des valeurs discrètes bien determiné. Ensuite, sous la condition d'égalité des vitesses de propagation ($a = 1$), nous montrons la non stabilité uniforme du système et nous établissons un taux de décroissance polynomial optimal de l'énergie du système dans un sous espace orthogonal à un sous espace de dimension finie explicitement connu. Enfin, quand les vitesses de propagation sont différentes et si $\sqrt{a}$ est un nombre rationnel, nous établissons un taux de décroissance polynomial de l'énergie (non nécessairement optimal).

Nous avons démontré le résultat général de la stabilité forte suivant :

Théorème 0.0.1. *(stabilité forte)*
Le sysème (0.0.1)-(0.0.2) est stable si et seulement si b satisfait les conditions suivantes (C_1) , (C_2) and (C_3) :

$$b \neq \frac{a}{a+1} 4k^2\pi^2, \forall k \in \mathbb{N}^*. \tag{C_1}$$

$$b \neq \frac{a(1-a)}{3a+1} 4k^2\pi^2, \forall k \in \mathbb{N}^*. \tag{C_2}$$

$$b \neq \frac{(ak_1^2 - k_2^2)(k_1^2 - ak_2^2)}{(a+1)(k_1^2 + k_2^2)} \pi^2, \forall k_1, k_2 \in \mathbb{N}^*, k_2 < k_1,$$
$$k_1, k_2 \text{ having the same parity.} \tag{C_3}$$

Notons que (C_2) est toujours vrai si $a \geq 1$.

D'abord, dans le cas $a = 1$, $b \neq 1$, et $b \neq 4\ell^2\pi^2$, $\ell \in \mathbb{Z}^*$, en utilisant une méthode spectrale, nous avons établissé le taux de décroissance de l'énergie suivant :

Théorème 0.0.2. *Supposons que* $a = 1$, $b \neq 1$, $b \neq 4\ell^2\pi^2$, $\ell \in \mathbb{Z}^*$, *et* b *satisfait les conditions* (C_1) *et* (C_3). *Alors il existe une constante* $C > 0$ *telle que pour tout* $U(0) = (u_0, u_1) \in D(\mathcal{A})$, *on a :*

$$E(t) \leq C \frac{\| U(0) \|^2_{D(\mathcal{A})}}{t}, \quad \forall t > 0. \tag{0.0.3}$$

Signalons que le taux polynomial obtenu est optimal.

Ensuite, dans le cas de différence de vitesse de propagation ($a \neq 1$) et si $\sqrt{a}$ est un nombre rationnel, en utilisant la méthode fréquentielle, nous avons obtenu la stabilité polynômiale du système (0.0.1)-(0.0.2) :

Théorème 0.0.3. *(décroissance polynômiale)*
Supposons que $a \neq 1$ *et* b *satisfait les conditions* (C_1) , (C_2) *and* (C_3). *En plus, supposons que* $\sqrt{a} \in \mathbb{Q}$, *alors il existe une constante positive* $c > 0$ *telle que pour toute condition initiale* $(u_0, u_1, y_0, y_1) \in D(\mathcal{A})$ *l'énergie du système (0.0.1)-(0.0.2) satisfait le taux de décroissance suivant :*

$$E(t) \leq \frac{c}{\sqrt[3]{t}} \|E(0)\|^2_{D(\mathcal{A})}, \quad \forall t > 0. \tag{0.0.4}$$

Chapitre 2. Taux de décroissance optimale de l'énergie de l'équation de poutre de Rayleigh avec un seul contrôle frontière force.
On considère l'équation de poutre de Rayleigh suivante :

$$\begin{aligned} y_{tt} - \gamma y_{xxtt} + y_{xxxx} &= 0, \ \ 0 < x < 1, \ \ t > 0, && (0.0.5) \\ y(0,t) = y_x(0,t) &= 0, \quad t > 0, && (0.0.6) \\ y_{xx}(1,t) + \alpha y_{xt}(1,t) &= 0, \quad t > 0, && (0.0.7) \\ y_{xxx}(1,t) - \gamma y_{xtt}(1,t) - \beta y_t(1,t) &= 0, \quad t > 0 && (0.0.8) \end{aligned}$$

où $\gamma > 0$ est le coefficient de moment d'inertie, $\beta > 0$ est le coefficient du contrôle force et $\alpha \geq 0$ est le coefficient du contrôle moment (dans ce Chapitre, nous considérons le cas d'une seule contrôle force *i.e.* $\alpha = 0$ et $\beta > 0$). Pour plus de détails concernant la modélisation du système, nous nous renvoyons à Russell [45].

Si $\gamma = 0$ l'équation de poutre de Rayleigh devient l'équation d'Euler-Bernoulli. Dans le cas d'un seul contrôle force $\alpha = 0$ et $\beta > 0$, la nature de stabilisation de l'équation de poutre de Rayleigh est différente de celle de l'équation d'Euler-Bernoulli. En effet, Chen *et al.* dans [18], [19] (voir aussi [30]) ont établi la stabilité uniforme de l'équation d'Euler-Bernoulli tandis que Rao dans [39], a prouvé la stabilité forte et non-uniforme de l'équation de poutre de Rayleigh si et seulement si le coefficient d'inertie γ est assez grand mais aucun taux de décroissance n'a été discuté. Alors, dans le cas d'un seul contrôle force $\alpha = 0$ et $\beta > 0$, la question sur le taux de décroissance de l'énergie et son optimalité reste un problème ouvert. Nous renvoyons le lecteur aux références [18], [19], [40], [41], [31] et [5] pour l'équation d'Euler-Bernoulli avec diffèrents types de mécanismes d'amortissement.

Différents types de contrôles ont été introduits à l'équation de poutre de Rayleigh et plusieurs résultats de stabilité ont été obtenus. Rao [42] a étudié la stabilisation de l'équation de poutre de Rayleigh avec un amortissement visqueux interne positif. En utilisant une approximation constructive, il a établi un taux optimal de décroissance exponentielle de l'énergie. Wehbe dans [50], a considéré l'équation de poutre de Rayleigh avec deux contrôles dynamiques frontières. Tout d'abord, en utilisant une méthode de perturbation compacte, il a prouvé que l'équation de poutre de Rayleigh n'est pas uniformément stable *i.e.* le taux de décroissance de l'énergie est non-exponentielle. Ensuite, en utilisant une méthode spectrale, il a établi le taux de décroissance optimal de l'énergie pour des données initiales régulières. Dans [27], Lagnese a étudié la stabilisation du système (0.0.5)-(0.0.8) avec deux contrôles frontières (le cas $\alpha > 0$ et $\beta > 0$). Il a prouvé que l'énergie décroît exponentiellement vers zéro pour toutes données initiales. Rao [39] a étendu les résultats de [27] au cas d'un seule contrôle frontière (le cas $\alpha > 0$, $\beta = 0$ ou $\alpha = 0$, $\beta > 0$). Dans le cas d'un seul contrôle moment (le cas $\alpha > 0$ et $\beta = 0$), en utilisant une théorie de perturbation compacte de Gibson [22], il a établi une stabilité exponentielle du système (0.0.5)-(0.0.8). De plus, dans le cas d'un seul contrôle force ($\alpha = 0$, $\beta > 0$), il a d'abord montré la non stabilité exponentielle du système (0.0.5)-(0.0.8). Ensuite, il a prouvé qu'il existe $\gamma_0 > 0$ tel que l'équation de poutre de Rayleigh est fortement stable si et seulement si le coefficient d'inertie $\gamma > \gamma_0$, mais aucun taux de décroissance n'a été discuté. Alors, dans ce cas, le taux de décroissance de l'énergie et son optimalité semble être un problème ouvert.

Dans ce Chapitre, nous considérons l'équation de poutre de Rayleigh avec un seul contrôle frontière ($\alpha = 0$, $\beta > 0$) dans le cas $\gamma > \gamma_0$. En utilisant une approximation explicite, nous donnons le développement asymptotique des valeurs propres et des fonctions propres du système non amorti correspondant au système (0.0.5)-(0.0.8). Ensuite, nous établissons un taux de décroissance polynômial de l'énergie pour des données initiales régulières via une inégalité d'observabilité du problème non amorti correspondant combiné avec la bornitude de la fonction de transfert du problème non amorti associé. Plus précisement nous avons obtenu le résultat suivant :

Théorème 0.0.4. *(décroissance polynômiale de l'énergie)*
Supposons que $\gamma > \gamma_0$, alors il existe une constante $c > 0$ telle que, pour tout $t > 0$ et pour tout $(y^0, y^1) \in D(\mathcal{A})$ la solution du système (0.0.5)-(0.0.8) vérifie l'estimation suivante :

$$E(y(t)) \leq \frac{c}{(1+t)} \|(y^0, y^1)\|^2_{D(\mathcal{A})}, \quad \forall t > 0. \tag{0.0.9}$$

Enfin, par une approche fréquentielle, en utilisant la partie réelle du comportement asymptotique des valeurs propres du générateur infinitésimal du semi-groupe associé, on montre que le taux de décroissance de l'énergie obtenu est optimal :

Théorème 0.0.5. *Le taux de décroissance de l'énergie (0.0.9) est optimal dans le sens où pour tout $\varepsilon > 0$, nous ne pouvons pas atteindre le taux de décroissance de type $\frac{1}{t^{1+\varepsilon}}$ pour toutes conditions initiales $U_0 \in D(\mathcal{A})$ et pour tout $t > 0$.*

Chapitre 3 . Certains résultats de stabilité sur l'équation Mindlin-Timoshenko dans un domaine non borné

Dans ce Chapitre, on considère la stabilisation interne de l'équation de la plaque de Mindlin-Timoshenko dans $\mathbb{R}^2$:

$$Jw_{tt} - K\text{div}(\nabla w + u) + bw_t = 0, \tag{0.0.10}$$

$$\rho u_{tt} - D(\frac{1-\mu}{2}\Delta u + \frac{1+\mu}{2}\nabla\text{div}u) + K(\nabla w + u) + au_t = 0, \tag{0.0.11}$$

dans $\mathbb{R}^2 \times (0, +\infty)$, avec les conditions initiales

$$u(x,0) = u^0(x), u_t(x,0) = u^1(x), w(x,0) = w^0(x), w_t(x,0) = w^1(x), \quad \forall x \in \mathbb{R}^2 \tag{0.0.12}$$

où J et ρ sont deux constantes qui dépendent de la masse par unité de surface et l'épaisseur uniforme de la plaque, K est le module de cisaillement, D est le module de rigidité en flexion et μ est le coefficient de Poisson ($0 < \mu < 1$ dans des situations physiques). La variable scalaire w représente le déplacement de la plaque dans la direction verticale, tandis que la variable vectorielle $u = (u_i)_{i=1}^2$ est l'angle de rotation d'un filament de la plaque (pour plus de détails, voir [26], [27]).

Dans [14], Belkacem et Kasimov ont étudié la stabilité monodimensionelle du système de Timoshenko dans $\mathbb{R}$ avec un seul amortissement de type Fourier ou Cattaneo. Ils ont montré que la dissipation de la chaleur seule est suffisante pour stabiliser le système dans les deux cas. Mais, il y a une différence considérable entre le système de Timeshenko dans $\mathbb{R}^2$ et celui dans $\mathbb{R}$. En effet, le couplage entre l'équation de déplacement transversal (0.0.10) et l'équation de la rotation angulaire (0.0.11) est donné par le gardient de la variable scalaire w et la variable vectorielle $u = (u_i)_{i=1}^2$, tandis que dans le cas monodimensionnel le couplage est donné par les dérivées partielles. Pour cette raison, les résultats de stabilité sont fondamentalement différentes. Alors, la question de stabiliser (0.0.10)-(0.0.12) reste un problème ouvert.

Dans ce chapitre, d'abord on considére un système de Mindlin-Timoshenko dans $\mathbb{R}^2$ avec deux lois de dissipations internes. En utilisant une approche directe fondée sur la transformation de Fourier, on montre que deux amortissements interne suffisent pour stabiliser polynomialement le système. Plus précisement, on montre :

Théorème 0.0.6. *Let* $U_0 = (w^0, w^1, u_1^0, u_2^0, u_1^1, u_2^1) \in \mathcal{H} \cap L^1(\mathbb{R}^2)^6$. *Alors, la solution* $U = (w, w', u_1, u_2, u_1', u_2')$ *du système (0.0.10)-(0.0.11) satisfait l'estimation suivante :*

$$\|U(t)\|_{\mathcal{H}}^2 \lesssim \|\widehat{U}_0\|_\infty^2 \frac{1}{t} + \|U_0\|_{\mathcal{H}}^2 e^{-ct},$$

avec $\|\widehat{U}_0\|_\infty = \|\widehat{u}_1^0\|_\infty + \|\widehat{u}_2^0\|_\infty + \|\widehat{v}_1^0\|_\infty + \|\widehat{v}_2^0\|_\infty + \|\widehat{w}^0\|_\infty + \|\widehat{y}^0\|_\infty$.

Ensuite, on considére un système de Mindlin-Timoshenko dans $\mathbb{R}^2$ avec deux lois de dissipations de type Fourier :

$$Jw_{tt} - K\text{div}(\nabla w + u) - \alpha\theta = 0, \qquad (0.0.13)$$

$$\widetilde{\rho}u_{tt} - D(\frac{1-\mu}{2}\Delta u + \frac{1+\mu}{2}\nabla\text{div}u) + K(\nabla w + u) + \delta\nabla\theta = 0, \qquad (0.0.14)$$

$$\theta_t - \Delta\theta + \delta\text{div}\, u_t + \alpha w_t = 0, \qquad (0.0.15)$$

dans $\mathbb{R}^2 \times (0, +\infty)$, avec les conditions initiales

$$u(x,0) = u^0(x),\ u_t(x,0) = u^1(x),\ w(x,0) = w^0(x),\ w_t(x,0) = w^1(x)\ \ x \in \mathbb{R}^2$$

$$\theta(x,0) = \theta^0(x), \quad \forall x \in \mathbb{R}^2. \qquad (0.0.16)$$

En utilisant une approche directe, on montre que le système (0.0.13)-(0.0.16) n'est pas stable :

Théorème 0.0.7. *Il existe* $U_0 \in \mathcal{H}$ *tel que l'énergie du système (0.0.13)-(0.0.16) reste constante , i.e*

$$E(t) = \frac{1}{2}\|U(x,t)\|^2 = E(0), \quad \forall t \geq 0. \tag{0.0.17}$$

Chapter 1

Polynomial stability of the Timoshenko system by one boundary damping

1.1 Introduction

In this chapter, we study the indirect boundary stabilization of the following Timoshenko system :

$$u_{tt} - (u_x + y)_x = 0 \quad \text{in } (0,1) \times (0,\infty), \tag{1.1.1}$$

$$y_{tt} - a y_{xx} + b(u_x + y) = 0 \quad \text{in } (0,1) \times (0,\infty), \tag{1.1.2}$$

$$u(0,t) = u(1,t) = y(1,t) = 0 \quad \text{in } (0,\infty), \tag{1.1.3}$$

with the following initial conditions

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x) \quad \text{in } (0,1) \tag{1.1.4}$$

and the boundary dissipation law

$$ay_x(0,t) = by_t(0,t) \qquad \text{in} \ \ (0,\infty), \tag{1.1.5}$$

where a and b are strictly positive constants. The functions u and y denote, respectively, the transverse displacement of the beam and the rotation angle of the filament. Let (u, y) be a regular solution of system (1.1.1)-(1.1.5), its associated total energy is defined by

$$E(t) = \frac{1}{2}\int_0^1 (|u_t|^2 + b^{-1}|y_t|^2 + ab^{-1}|y_x|^2 + |u_x + y|^2)dx. \tag{1.1.6}$$

Then a straightforward computation gives

$$\frac{d}{dt}E(t) = -|y_t(0,t)|^2 \leq 0. \tag{1.1.7}$$

Hence system (1.1.1)-(1.1.5) is dissipative in the sense that its associated energy is non increasing with respect to time.

The stabilization of the Timoshenko system has been studied with different types of dampings. For the internal stabilization, Raposo *et al.* [44] studied the stabilization of the Timoshenko system with two internal distributed dissipations and proved that the energy decays exponentially. Messaoudi and Mustafa [34] extended the results to nonlinear feedback laws. Soufyane and Wehbe in [48] considered the system (1.1.1)-(1.1.4) with one internal distributed dissipation law and they proved that the uniform stability holds if and only if the wave speeds are equal ($a = 1$); otherwise only the asymptotic stability has been proved. This result has been recently improved by Rivera and Racke [35], where an exponential decay of the solution of the system has been established, allowing the coefficient of the feedback to be with an indefinite sign. Also, Rivera and Racke [36] considered a nonlinear Timoshenko system with one linear internal distributed feedback. They proved that the system is exponentially stable if and only if the wave propagation speeds are equal. Otherwise, only the polynomial stability holds. Alabau-Boussouira [8] extended the results of [36] to the case of nonlinear feedback $\alpha(\psi_t)$, where α is a globally Lipchitz function satisfying some growth conditions at the origin. Wehbe and Youssef [51] considerd the Timoshenko system with one locally distributed feedback. They proved that the system is exponentially stable if and only if the wave propagation speeds are equal. Otherwise, only the polynomial stability holds. For the stabilization of the system with memory or past history term, see [9], [32], [33].

Now, let us mention some know results related to the boundary stabilization of the Timoshenko beam. Kim and Renardy in [25] proved the exponential stability of the system under two boundary controls. In [10], Ammar-Khodja and al. studied the decay rate of the energy of the nonuniform Timoshenko beam with two boundary controls acting in the rotation-angle equation. Under the equal speed wave propagation condition, they established exponential decay results up to a unknown finite dimensional space of initial data. In addition, they showed that the equal speed wave propagation condition is necessary for the exponential stability. However, in the case of non equal speed, no decay rate has been discussed.

In this paper, we study the decay rate of the energy of the Timoshenko beam with one boundary control acting in the rotation-angle equation. Under the equal speed condition ($a = 1$) and if b is outside a discrete set of exceptional values, using a spectral analysis, we prove non uniform stability but optimal polynomial energy decay rate is obtained. On the other hand if $\sqrt{a}$ is a rational number and if b is outside another discrete set of exceptional values, we also show a polynomial type decay rate using a frequency domain approach. Note that, contrary to what we announced in [12], the exponential decay does not hold in the case $a = 1$.

1.2 Well-posedness and strong stability.

In this section we study the existence, uniqueness and strong stability of the solution of system (1.1.1)-(1.1.5). Let us set

$$\Omega = (0,1) \quad \text{and} \quad H_R^1(\Omega) = \{y \in H^1(\Omega) : y(1) = 0\}.$$

Define the energy space $\mathcal{H}$ as follows

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times H_R^1(\Omega) \times L^2(\Omega),$$

with the inner product defined by

$$\Big(U, U_1\Big)_{\mathcal{H}} = \int_0^1 \Big(v\overline{v_1} + b^{-1}z\overline{z_1} + ab^{-1}y_x\overline{y_{1x}} + (u_x + y)(\overline{u_{1x} + y_1})\Big)dx, \tag{1.2.1}$$

for all $U = (u, v, y, z)$, $U_1 = (u_1, v_1, y_1, z_1) \in \mathcal{H}$.

Remark 1.2.1. The norm $\Big(U, U\Big)_{\mathcal{H}}^{\frac{1}{2}}$ induced by (1.2.1) is equivalent to the usual norm of $\mathcal{H}$.

For shortness we denote by $\|\cdot\|$ the $L^2(\Omega)$-norm.

Now, we define a linear unbounded operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ by:

$$D(\mathcal{A}) = \{U \in \mathcal{H} : u, y \in H^2(\Omega), v \in H_0^1(\Omega), z \in H_R^1(\Omega), ay_x(0) - bz(0) = 0\}, \tag{1.2.2}$$

$$\mathcal{A}(u, v, y, z) = \big(v, (u_x+y)_x, z, ay_{xx} - b(u_x+y)\big), \quad \forall U = (u, v, y, z) \in D(\mathcal{A}). \tag{1.2.3}$$

Then we rewrite formally system (1.1.1)-(1.1.5) into the evolution equation

$$\begin{cases} U_t = \mathcal{A}U, \\ U(0) = U_0, \qquad U_0 \in \mathcal{H}, \end{cases} \tag{1.2.4}$$

with $U = (u, u_t, y, y_t)$.

Proposition 1.2.2. *The operator $\mathcal{A}$ is m-dissipative in the energy space $\mathcal{H}$.*

Proof. We start with the dissipativeness.
Let $U = (u, v, y, z) \in D(\mathcal{A})$. Using (1.2.1) and (1.2.3), we obtain :

$$(\mathcal{A}U, U)_{\mathcal{H}} = \int_0^1 \Big((u_x+y)_x\overline{v}+b^{-1}\Big(ay_{xx}-b(u_x+y)\Big)\overline{z}+ab^{-1}z_x\overline{y_x}+(v_x+z)(\overline{u_x+y})\Big)dx.$$

Then, integrating by parts and using the boundary conditions, we get

$$Re(\mathcal{A}U, U)_{\mathcal{H}} = -|z(0)|^2 \leq 0. \tag{1.2.5}$$

Let us pass to the maximality.
Let $f = (f_1, f_2, f_3, f_4) \in \mathcal{H}$, we look for a unique element $U = (u, v, y, z) \in D(\mathcal{A})$ such that

$$-\mathcal{A}U = f.$$

Equivalently, we get $v = -f_1$, $z = -f_3$ and the following system

$$-(u_x + y)_x = f_2 \qquad \text{in } (0, 1), \tag{1.2.6}$$

$$-ay_{xx} + b(u_x + y) = f_4 \qquad \text{in } (0, 1). \tag{1.2.7}$$

Assuming that such a solution (u, y) exists. Then multiplying (1.2.7) (resp. (1.2.6)) by $\bar{y}_1 \in H^1_R(\Omega)$ (resp by $b\bar{u}_1$ with $u_1 \in H^1_0(\Omega)$), integrating in Ω and taking the sum, we obtain

$$\int_\Omega (-ay_{xx}\bar{y}_1 + b(u_x + y)\bar{y}_1 - b(u_x + y)_x\bar{u}_1)\ dx = \int_\Omega (f_4\bar{y}_1 + bf_2\bar{u}_1)\ dx.$$

Two integrations by parts and taking into account the boundary condition $ay_x(0) = bz(0) = -bf_3(0)$, we obtain

$$c((u, y), (u_1, y_1)) = \int_\Omega (f_4\bar{y}_1 + bf_2\bar{u}_1)\ dx + bf_3(0)\bar{y}_1(0), \forall (u_1, y_1) \in H^1_0(\Omega) \times H^1_R(\Omega), \tag{1.2.8}$$

where

$$c((u,y),(u_1,y_1)) = \int_\Omega (ay_x\bar{y}_{1x} + b(u_x+y)(\bar{u}_{1x}+\bar{y}_1))\ dx.$$

Since

$$c((u,y),(u,y)) = \int_\Omega \left(a|y_x|^2 + b|u_x+y|^2\right)\ dx,$$

the sesquilinear form c is strongly coercive on $H_0^1(\Omega)\times H_R^1(\Omega)$, and by Lax-Milgram lemma, problem (1.2.8) admits a unique solution $(u,y)\in H_0^1(\Omega)\times H_R^1(\Omega)$. By taking test functions in the form $(\varphi,0)$ and $(0,\psi)$ with $\varphi,\psi\in\mathcal{D}(\Omega)$, it is easy to see that (u,y) satisfies system (1.2.6)-(1.2.7) in the distributional sense. This also shows that u and y belong to $H^2(\Omega)$ because

$$u_{xx} = -f_2 + y_x \in L^2(\Omega),$$

and

$$ay_{xx} = -f_4 + b(u_x+y) \in L^2(\Omega).$$

Coming back to (1.2.8) and integrating by parts we find that

$$ay_x(0) = -bf_3(0).$$

Setting $v=-f_1$, $z=-f_3$ we have shown that (u,v,y,z) belongs to $D(\mathcal{A})$ and is a solution of $-\mathcal{A}U=f$. Therefore we deduce that $0\in\rho(\mathcal{A})$. Then by the resolvent identity, for $\lambda>0$ small enough we have $R(\lambda I-\mathcal{A})=\mathcal{H}$ (see Theorem 1.2.4 in [29]). The proof is thus complete.

Using Lumer-Phillips Theorem (see [38], Theorem 1.4.3), the operator $\mathcal{A}$ generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ on $\mathcal{H}$. Then, we have the following results.

Theorem 1.2.3. *(Existence and uniqueness)*
(1) If $U_0\in D(\mathcal{A})$, then system (1.2.4) has a unique strong solution

$$U\in C^0(\mathbb{R}_+, D(\mathcal{A}))\cap C^1(\mathbb{R}_+,\mathcal{H}).$$

(2) If $U_0 \in \mathcal{H}$, then system (1.2.4) has a unique weak solution

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

Now, we will prove the following general strong stability result

Theorem 1.2.4. *(Strong stability)*
The system (1.1.1)-(1.1.5) is strongly stable if and only if the coefficient b satisfies the following conditions (C_1), (C_2) and (C_3) :

$$b \neq \frac{a}{a+1} 4k^2\pi^2, \forall k \in \mathbb{N}^*. \qquad (C_1)$$

$$b \neq \frac{a(1-a)}{3a+1} 4k^2\pi^2, \forall k \in \mathbb{N}^*. \qquad (C_2)$$

$$b \neq \frac{(ak_1^2 - k_2^2)(k_1^2 - ak_2^2)}{(a+1)(k_1^2 + k_2^2)}\pi^2, \forall k_1, k_2 \in \mathbb{N}^*, k_2 < k_1,$$

$$k_1, k_2 \text{ having the same parity.} \qquad (C_3)$$

Note that (C_2) always holds if $a \geq 1$.

Proof: Since $\mathcal{A}$ generates a contraction semigroup and its resolvent is compact in $\mathcal{H}$, using Arendt-Batty theorem (see [11, p. 837]), system (1.1.1)-(**??**) is strongly stable if and only if $\mathcal{A}$ does not have pure imaginary eigenvalues. It then remains to prove that this last condition is equivalent with the conditions $(C_i), i = 1, .., 3$.

Let $\lambda \in \mathbb{R} \setminus \{0\}$ be such that $i\lambda \in \sigma(\mathcal{A})$. Then, there exists $U = (u, v, y, z) \in D(\mathcal{A}) \setminus \{0\}$ such that

$$\mathcal{A}U = i\lambda U.$$

Hence using the dissipativeness of $\mathcal{A}$ (see (**??**)), we get

$$|z(0)|^2 = -Re(\mathcal{A}U, U)_{\mathcal{H}} = 0.$$

This means that $z(0) = 0$ and by the definition of the domain of $\mathcal{A}$, we find that

$$y_x(0) = 0.$$

Coming back to the definition of $\mathcal{A}$, we find that (u, y) satisfies

$$\begin{cases} u_{xx} + y_x + \lambda^2 u &= 0, \\ a y_{xx} - b u_x - b y + \lambda^2 y &= 0, \end{cases} \tag{1.2.9}$$

$$\begin{cases} u(0) = y_x(0) = y(0) = 0, \\ u(1) = y(1) = 0. \end{cases} \tag{1.2.10}$$

Let $\mu = \lambda^2 > 0$. First, we search a basis of fundamental solutions of (1.2.9). For that purpose, we look for a solution (u, y) of the form $e^{tx}(w_1, w_2)$. We see that (w_1, w_2) is solution of the system

$$\begin{pmatrix} t^2 + \mu & t \\ -bt & at^2 - b + \mu \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{1.2.11}$$

The determinant of this system is

$$P(t) = at^4 + Bt^2 + C$$

where $B = (a+1)\mu$ and $C = \mu^2 - b\mu$. Setting $P_1(m) = am^2 + Bm + C$. The discriminant of P_1 is :

$$\Delta = B^2 - 4aC = \mu^2(a-1)^2 + 4ab\mu.$$

We have $\Delta \geq 4ab\mu > 0$, then the polynomial P_1 has two distinct real roots m_1 and m_2 given by:

$$m_1 = \frac{-B - \sqrt{\Delta}}{2a} \text{ and } m_2 = \frac{-B + \sqrt{\Delta}}{2a}.$$

It is clear that $m_1 < 0$. As $B^2 - \Delta = -4a\mu(b - \mu)$, the sign of m_2 depends on the value of μ with respect to b. Therefore, we distinguish the three cases: $\mu < b$, $\mu = b$ and $\mu > b$.

Case 1: $\mu < b$ (thus $m_2 > 0$).
Setting

$$t_1 = \sqrt{-m_1} \text{ and } t_2 = \sqrt{m_2}. \tag{1.2.12}$$

Then P has 4 roots $it_1, -it_1, t_2, -t_2$, and after easy calculations we find that the general solution of (1.2.9) is

$$u(x) = a_1 \sin(t_1 x) + a_2 \cos(t_1 x) + a_3 \sinh(t_2 x) + a_4 \cosh(t_2 x), \qquad (1.2.13)$$

$$y(x) = a_1 d_1 \cos(t_1 x) - a_2 d_1 \sin(t_1 x) + a_3 d_2 \cosh(t_2 x) + a_4 d_2 \sinh(t_2 x), \qquad (1.2.14)$$

where $a_j, j = 1, \ldots, 4$ are complex numbers and d_1 and d_2 are given by

$$d_1 = \frac{\mu}{t_1} - t_1 \text{ and } d_2 = -\frac{\mu}{t_2} - t_2. \qquad (1.2.15)$$

Now, we search for $(u, y) \neq (0, 0)$ satisfying the boundary conditions (2.2.10). First $u(0) = 0$ implies $a_4 = -a_2$. Second we have $y_x(0) = -a_2(d_1 t_1 + d_2 t_2) = 0$. But

$$d_1 t_1 + d_2 t_2 = -t_1^2 - t_2^2 = m_1 - m_2 = -\frac{\sqrt{\Delta}}{a} \neq 0. \qquad (1.2.16)$$

Therefore

$$a_2 = a_4 = 0.$$

On the other hand, the conditions $u(1) = y(1) = 0$ is equivalent to

$$\begin{pmatrix} \sin(t_1) & \sinh(t_2) \\ d_1 \cos(t_1) & d_2 \cosh(t_2) \end{pmatrix} \begin{pmatrix} a_1 \\ a_3 \end{pmatrix} = 0. \qquad (1.2.17)$$

Since we assume that $(u, y) \neq (0, 0)$, we deduce that the determinant of system (1.2.17) vanishes :

$$-\cos(t_1) \sinh(t_2) d_1 + \sin(t_1) \cosh(t_2) d_2 = 0. \qquad (1.2.18)$$

In that case, it is easy to see that the rank of the matrix in (1.2.17) is one. Then by setting

$$a_1 = \sinh(t_2) \qquad \text{and} \quad a_3 = -\sin(t_1),$$

we get as eigenvectors:

$$u(x) = \sinh(t_2) \sin(t_1 x) - \sin(t_1) \sinh(t_2 x),$$

$$y(x) = \sinh(t_2)d_1\cos(t_1x) - \sin(t_1)d_2\cosh(t_2x).$$

Combining the boundary condition $y(0) = 0$ with the characteristic equation (1.2.18), we see that (d_1, d_2) is solution of a homogeneous system, and, since from (1.2.16) we know that $(d_1, d_2) \neq (0, 0)$, then the determinant of this system vanishes. Hence, we have:

$$\sin(t_1)\sinh(t_2)(\cosh(t_2) - \cos(t_1)) = 0.$$

But from (1.2.16) this is not possible. Consequently, in the case 1, (1.2.9)-(1.2.10) admits only the trivial solution.

<u>Case 2</u>: $\mu = b$ (thus $m_2 = 0$).

Similarly, let

$$t_1 = \sqrt{-m_1} = \sqrt{\frac{(1+a)b}{a}}. \tag{1.2.19}$$

Then P has 2 simple roots $it_1, -it_1$, and 0 as a double root. With a simple calculation, we find the following general solution of (1.2.9)

$$u(x) = a_1\sin(t_1x) + a_2\cos(t_1x) + a_3, \tag{1.2.20}$$

$$y(x) = a_1d_1\cos(t_1x) - a_2d_1\sin(t_1x) - a_3bx + a_4, \tag{1.2.21}$$

where $a_j, j = 1, \ldots, 4$ are complex numbers and $d_1 = \dfrac{\mu}{t_1} - t_1 = \dfrac{b}{t_1} - t_1$. We can directly see that a solution (u, y) of (1.2.9) defined by (1.2.20)-(1.2.21) and satisfying

$$\begin{cases} u(0) = y_x(0) = 0, \\ u(1) = y(1) = 0, \end{cases}$$

is non-trivial if and only if

$$\sin(t_1) = 0.$$

In that case the set of solution is of dimension 1 and a basis is :

$$\begin{cases} u(x) = \sin(t_1 x), \\ y(x) = d_1 \cos(t_1 x) - d_1 \cos(t_1). \end{cases}$$

Moreover, the boundary condition $y(0) = 0$ is satisfied if $\cos(t_1) = 1$. Consequently, we deduce that under the condition (C_1), the problem (2.2.9)-(2.2.10) has only a trivial solution. Conversely, if (C_1) is not valid, we have found a non-trivial solution of problem (1.2.9)-(1.2.10).

Case 3: $\mu > b$ (thus $m_2 < 0$). Setting

$$t_1 = \sqrt{-m_1} \text{ and } t_2 = \sqrt{-m_2}. \tag{1.2.22}$$

Then P has four roots $it_1, -it_1, it_2, -it_2$, and the general solution of (1.2.9) is

$$u(x) = a_1 \sin(t_1 x) + a_2 \cos(t_1 x) + a_3 \sin(t_2 x) + a_4 \cos(t_2 x), \tag{1.2.23}$$

$$y(x) = a_1 d_1 \cos(t_1 x) - a_2 d_1 \sin(t_1 x) + a_3 d_2 \cos(t_2 x) - a_4 d_2 \sin(t_2 x) \tag{1.2.24}$$

where $a_j, j = 1, \ldots, 4$ are complex numbers and $d_i, i = 1, 2$ are given by (**??**). Using the conditions $u(0) = 0$ we get $a_4 = -a_2$. Therefore the condition $y_x(0) = 0$ becomes

$$-a_2(d_1 t_1 - d_2 t_2) = -a_2(-t_1^2 + t_2^2) = a_2(-m_1 + m_2) = 0.$$

Since $m_1 \neq m_2$ we get $a_2 = a_4 = 0$. Therefore (u, y) is in the form:

$$\begin{cases} u(x) = a_1 \sin(t_1 x) + a_3 \sin(t_2 x), \\ y(x) = a_1 d_1 \cos(t_1 x) + a_3 d_2 \cos(t_2 x). \end{cases}$$

The boundary conditions $u(1) = y(1) = y(0) = 0$ imply that (a_1, a_3) is solution of the following system :

$$\begin{pmatrix} \sin(t_1) & \sin(t_2) \\ d_1 \cos(t_1) & d_2 \cos(t_2) \\ d_1 & d_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_3 \end{pmatrix} = 0.$$

Since we assume that $(u, y) \neq (0, 0)$ then $(a_1, a_3) \neq (0, 0)$. Moreover, it is easy to see that $d_1 \neq 0$ and $d_2 \neq 0$. Therefore the rank of the previous matrix is one.
That means that we have $\cos(t_1) = \cos(t_2)$, $\sin(t_2) = \epsilon sin(t_1)$ with $\epsilon = \pm 1$, and $\sin(t_1)(d_2 - \epsilon d_1) = 0$.
First, assume that $\sin(t_1) = 0$. Then $\sin(t_2) = 0$ and there exist $k_1, k_2 \in \mathbb{N}^*, k_1 > k_2$, (remember that $t_1 > t_2 > 0$) with the same parity such that $t_1 = k_1\pi$ and $t_2 = k_2\pi$.
Hence

$$(k_1^2 + k_2^2)\pi^2 = t_1^2 + t_2^2 = -m_1 - m_2 = \frac{B}{a} = \frac{a+1}{a}\mu.$$

We have also

$$k_1^2 k_2^2 \pi^4 = t_1^2 t_2^2 = m_1 m_2 = \frac{c}{a} = \frac{\mu^2 - b\mu}{a}.$$

Eliminating μ we find that

$$b = \frac{(ak_1^2 - k_2^2)(k_1^2 - ak_2^2)}{(a+1)(k_1^2 + k_2^2)}\pi^2.$$

Hence under this condition (i.e if (C_3) does not hold), the choice $(a_1, a_3) = (-d_2 cos(t_2), d_1 cos(t_1))$ leads to a non-trivial solution of (1.2.9)-(1.2.10). Conversely if (C_3) holds, λ is not an eigenvalue of $\mathcal{A}$.
Second, assume that $\sin(t_1) \neq 0$, then $d_2 = \epsilon d_1$ with $\epsilon = \pm 1$.
If $\epsilon = 1$ then $d_1 = d_2$, which is equivalent with $t_1 t_2 = -\mu$, which is not possible since $t_1 > 0$ and $t_2 > 0$.
If $\epsilon = -1$ then $d_1 = -d_2$, which is equivalent with $t_1 t_2 = \mu$, or again with $m_1 m_2 = \dfrac{\mu^2 - b\mu}{a} = \mu^2$. This last equality is possible only if $a \neq 1$ and in that case $\mu = \dfrac{b}{1-a}$. Remark also that $\mu > b$ thus $a < 1$.
Now, since $\cos(t_1) = \cos(t_2)$ and $\sin(t_1) = -\sin(t_2)$, there exists $k \in \mathbb{N}^*$ such that $t_1 + t_2 = 2k\pi$. A computation shows that $(t_1 + t_2)^2 = \dfrac{(1+3a)b}{(1-a)a}$. Hence if $\dfrac{(1+3a)b}{(1-a)a} = 4k^2\pi^2$ for some $k \in \mathbb{N}^*$ (i.e if (C_2) does not hold), the choice $(a_1, a_3) = (1, 1)$ yields a non-trivial solution of (2.2.9)-(2.2.10).
Conversely, if (C_2) holds, then (1.2.9)-(1.2.10) has only the trivial solution.

Remark 1.2.5. If the coefficient b does not satisfy one of the conditions (C_1), (C_2) or (C_3), the operator $\mathcal{A}$ has a finite number of purely imaginary eigenvalues with explicit eigenvectors. In that case we can show the strong and polynomial stability in the space orthogonal to these eigenvectors that is invariant under the action of $\mathcal{A}$.

Remark 1.2.6. Assume that $a = 1$. Then the system (1.1.1)-(1.1.5) is strongly stable if and only if the coefficient b satisfies the conditions (C_1) and (C_3).

1.3 Non uniform stability result

In this section we show that uniform stability (i.e exponential stability) does not hold even if the wave speeds are assumed to be equal. This result is due to the fact that a subsequence of eigenvalues of $\mathcal{A}$ is close to the imaginary axis. We first consider the case $a = 1$.

1.3.1 The case $a = 1, b \neq 4\ell^2\pi^2, \ell \in \mathbb{N}^*$

We first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$.

Let λ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (u, v, y, z)$. Then $\mathcal{A}U = \lambda U$ is equivalent to

$$\begin{cases} v = \lambda u, \\ u_{xx} + y_x = \lambda v, \\ z = \lambda y, \\ y_{xx} - bu_x - by = \lambda z, \\ u(0) = u(1) = y(1) = y_x(0) - bz(0) = 0. \end{cases} \tag{1.3.1}$$

Eliminating v and z we get

$$\begin{cases} (i) & u_{xx} + y_x - \lambda^2 u = 0, \\ (ii) & y_{xx} - bu_x - by - \lambda^2 y = 0, \\ (iii) & u(0) = u(1) = y(1) = y_x(0) - b\lambda y(0) = 0. \end{cases}$$

Now, from (i) we have $y_x = -u_{xx} + \lambda^2 u$, thus after derivation of (ii) we get

$$(-u_{xx} + \lambda^2 u)_{xx} - bu_{xx} - (b + \lambda^2)(-u_{xx} + \lambda^2 u) = 0$$

or

$$u_{xxxx} - 2\lambda u_{xx} + (b + \lambda^2)\lambda^2 u = 0. \tag{1.3.2}$$

Then, we write (iii) uniquely in function of u.

From (i) and (iii) we have

$$y_x(0) = \lambda^2 u(0) - u_{xx}(0) = -u_{xx}(0),$$

and from (ii) we have

$$\frac{y_{xx} - bu_x}{b + \lambda^2} = y(x).$$

We have also

$$y_{xx} = -u_{xxx} + \lambda^2 u_x.$$

Thus we find

$$y(x) = \frac{-u_{xxx} + (\lambda^2 - b)u_x}{b + \lambda^2}. \tag{1.3.3}$$

Consequently, the third condition of (iii) becomes

$$-u_{xxx}(1) + (\lambda^2 - b)u_x(1) = 0, \tag{1.3.4}$$

and the fourth condition of (iii) becomes (after multiplication by $b + \lambda^2$)

$$-(b + \lambda^2)u_{xx}(0) + b\lambda(u_{xxx}(0) + (b - \lambda^2)u_x(0)) = 0. \tag{1.3.5}$$

Finally we have found that λ is an eigenvalue of $\mathcal{A}$ (except may be $\lambda = \pm i\sqrt{b}$) if and only if there is a non trivial solution of (1.3.2) which satisfies the first and second boundary condition of (iii) and the boundary conditions (1.3.4) and (1.3.5).

The general solution of (1.3.2) is given by

$$u(x) = \sum_{i=1}^{4} c_i e^{t_i x}, \tag{1.3.6}$$

where $t_1(\lambda) = \sqrt{\lambda^2 + i\sqrt{b}\lambda}, t_2(\lambda) = -t_1(\lambda), t_3(\lambda) = \sqrt{\lambda^2 - i\sqrt{b}\lambda}$ and $t_4(\lambda) = -t_3(\lambda)$.

Here and below, for simplicity we denote $t_i(\lambda)$ by t_i .

Thus the boundary conditions may be written as the following system:

$$M(\lambda)C(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ e^{t_1} & e^{t_2} & e^{t_3} & e^{t_4} \\ h_{1,\lambda}(t_1)e^{t_1} & h_{1,\lambda}(t_2)e^{t_2} & h_{1,\lambda}(t_3)e^{t_3} & h_{1,\lambda}(t_4)e^{t_4} \\ h_{2,\lambda}(t_1) & h_{2,\lambda}(t_2) & h_{2,\lambda}(t_3) & h_{2,\lambda}(t_4) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0,$$

where we have set $h_{1,\lambda}(t) = -t^3 + (\lambda^2 - b)t$ and $h_{2,\lambda}(t) = -(b+\lambda^2)t^2 + b\lambda(t^3 + (b - \lambda^2)t)$. Hence a non trivial solution u exists if and only if the determinant of $M(\lambda)$ vanishes. Set $f(\lambda) = detM(\lambda)$, thus the characteristic equation is $f(\lambda) = 0$.

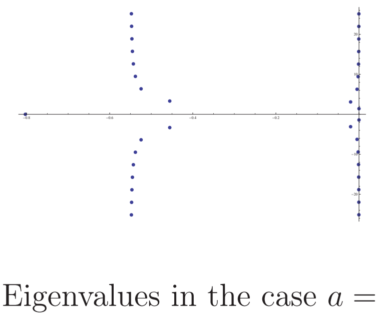

Figure 1.1: Eigenvalues in the case $a = 1$ and $b = 2$

A numerical distribution of the roots of f (or equivalently the spectrum of $\mathcal{A}$) in the case $a = 1$ and $b = 2$ is given in Figure 1.1. We see that a part of the spectrum of $\mathcal{A}$ seems to be very close to the imaginary axis. Hence, our purpose in the sequel is to prove, thanks to Rouché's theorem, that effectively there is a subsequence of eigenvalues for which their real part tends to 0.

In the sequel, since $\mathcal{A}$ is dissipative, we study the asymptotic behavior of the large eigenvalues λ of $\mathcal{A}$ in the strip $-\alpha_0 \leq \Re(\lambda) \leq 0$, for some $\alpha_0 > 0$ large enough and for such λ, we remark that $e^{t_i}, i = 1, \ldots, 4$ remains bounded.

Lemma 1.3.1. *Assume that $b \neq 1$ and $b \neq 4\ell^2\pi^2, \ell \in \mathbb{N}^*$. There exists $N \in \mathbb{N}$ such that*

$$\{\lambda_k\}_{k\in\mathbb{Z}^*,|k|\geq N} \cup \{\mu_k\}_{k\in\mathbb{Z},|k|\geq N} \subset \sigma(\mathcal{A}), \tag{1.3.7}$$

where

- $\lambda_k = ik\pi + \dfrac{\alpha}{k} + \dfrac{\beta}{k^2} + o(\dfrac{1}{k^2}), k \in \mathbb{Z}^*, |k| \geq N, \alpha \in i\mathbb{R}$ and $\beta \in \mathbb{R}, \beta < 0$,
- $\mu_k = \dfrac{1}{2}\ln\dfrac{b-1}{b+1} + ik\pi + o(1), k \in \mathbb{Z}$ and $|k| \geq N$.

Moreover for all $|k| \geq N$, the eigenvalues λ_k and μ_k are simple.

Proof. The proof is decomposed in four steps :
Step 1. For $b < 1$, we choose $\dfrac{1}{2}\ln\dfrac{b-1}{b+1} = \dfrac{1}{2}\ln|\dfrac{b-1}{b+1}| + i\dfrac{\pi}{2}$.

We start by the expansion of t_1 and t_3 :

$$t_1(\lambda) = \lambda + \frac{i\sqrt{b}}{2} + \frac{b}{8\lambda} - \frac{ib\sqrt{b}}{16\lambda^2} + O(\frac{1}{\lambda^3}), \tag{1.3.8}$$

$$t_3(\lambda) = \lambda - \frac{i\sqrt{b}}{2} + \frac{b}{8\lambda} + \frac{ib\sqrt{b}}{16\lambda^2} + O(\frac{1}{\lambda^3}). \tag{1.3.9}$$

Using (1.3.8) and (1.3.9), we find the asymptotic development of :

$$h_{1,\lambda}(t_1) = -i\sqrt{b}\lambda^2 - \frac{b}{2}\lambda - \frac{5ib\sqrt{b}}{8} + O(\frac{1}{\lambda}), \tag{1.3.10}$$

$$h_{1,\lambda}(t_2) = i\sqrt{b}\lambda^2 + \frac{b}{2}\lambda + \frac{5ib\sqrt{b}}{8} + O(\frac{1}{\lambda}), \tag{1.3.11}$$

$$h_{1,\lambda}(t_3) = i\sqrt{b}\lambda^2 - \frac{b}{2}\lambda + \frac{5ib\sqrt{b}}{8} + O(\frac{1}{\lambda}), \tag{1.3.12}$$

$$h_{1,\lambda}(t_4) = -i\sqrt{b}\lambda^2 + \frac{b}{2}\lambda - \frac{5ib\sqrt{b}}{8} + O(\frac{1}{\lambda}), \tag{1.3.13}$$

and,

$$h_{2,\lambda}(t_1) = -\lambda^4 - \lambda^3 i\sqrt{b}(1-b) - b\lambda^2(1-\frac{b}{2}) - ib\sqrt{b}\lambda(1-\frac{5b}{8}) + O(1), \tag{1.3.14}$$

$$h_{2,\lambda}(t_2) = -\lambda^4 - \lambda^3 i\sqrt{b}(1+b) - b\lambda^2(1+\frac{b}{2}) - ib\sqrt{b}\lambda(1+\frac{5b}{8}) + O(1), \tag{1.3.15}$$

$$h_{2,\lambda}(t_3) = -\lambda^4 + \lambda^3 i\sqrt{b}(1-b) - b\lambda^2(1-\frac{b}{2}) + ib\sqrt{b}\lambda(1-\frac{5b}{8}) + O(1), \tag{1.3.16}$$

$$h_{2,\lambda}(t_4) = -\lambda^4 + \lambda^3 i\sqrt{b}(1+b) - b\lambda^2(1+\frac{b}{2}) + ib\sqrt{b}\lambda(1+\frac{5b}{8}) + O(1). \tag{1.3.17}$$

Multiply the third line of $M(\lambda)$ by $-\frac{1}{4i\sqrt{b}\lambda^2}$ and the fourth line by $-\frac{1}{\lambda^3}$, we obtain an equivalent system for $M(\lambda)C(\lambda) = 0$ given by

$$\widetilde{M}(\lambda)C(\lambda) = 0, \tag{1.3.18}$$

where,

$$\widetilde{M}(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ e^{t_1} & e^{t_2} & e^{t_3} & e^{t_4} \\ -\frac{1}{4i\sqrt{b}\lambda^2}h_{1,\lambda}(t_1)e^{t_1} & -\frac{1}{4i\sqrt{b}\lambda^2}h_{1,\lambda}(t_2)e^{t_2} & -\frac{1}{4i\sqrt{b}\lambda^2}h_{1,\lambda}(t_3)e^{t_3} & -\frac{1}{4i\sqrt{b}\lambda^2}h_{1,\lambda}(t_4)e^{t_4} \\ -\frac{1}{\lambda^3}h_{2,\lambda}(t_1) & -\frac{1}{\lambda^3}h_{2,\lambda}(t_2) & -\frac{1}{\lambda^3}h_{2,\lambda}(t_3) & -\frac{1}{\lambda^3}h_{2,\lambda}(t_4) \end{pmatrix}. \tag{1.3.19}$$

Using the asymptotic development (1.3.8)-(1.3.17) and after some computations, we find the following asymptotic development of $\tilde{f}(\lambda)$ the determinant of $\widetilde{M}(\lambda)$,

$$\tilde{f}(\lambda) = f_0(\lambda) + \frac{f_1(\lambda)}{\lambda} + \frac{f_2(\lambda)}{\lambda^2} + O(\frac{1}{\lambda^3}), \tag{1.3.20}$$

where

$$f_0(\lambda) = 4e^{-t_1-t_3}(e^{t_1+t_3} - 1)(1 - b + (1+b)(e^{t_1+t_3})), \tag{1.3.21}$$

$$f_1(\lambda) = 2b(e^{t_1-t_3} - e^{t_3-t_1}), \tag{1.3.22}$$

and,

$$f_2(\lambda) = 12ib^2\sqrt{b} - \frac{13ib\sqrt{b}}{2}(e^{t_1+t_3} - e^{-t_1-t_3}) - 5ib^2\sqrt{b}(e^{t_1+t_3} + e^{-t_1-t_3}) - ib^2\sqrt{b}(e^{t_3-t_1} + e^{t_1-t_3}). \tag{1.3.23}$$

Note that f_0, f_1 and f_2 remains bounded in the strip $-\alpha_0 \leq \Re(\lambda) \leq 0$.

Step 2. We look at the roots of f_0. From (1.3.21), f_0 has two families of roots that we denote λ_k^0 and μ_k^0.

First case : $e^{t_1+t_3} = 1$ or equivalently

$$t_1 + t_3 = 2ik\pi, k \in \mathbb{Z},$$

i.e.,

$$\sqrt{\lambda^2 + i\sqrt{b}\lambda} + \sqrt{\lambda^2 - i\sqrt{b}\lambda} = 2ik\pi, k \in \mathbb{Z}.$$

This together with the assumption $b \neq 4\ell^2\pi^2, \ell \in \mathbb{N}^*$, yield

$$\lambda = \pm i \frac{2k^2\pi^2}{\sqrt{4k^2\pi^2 - b}}, k \in \mathbb{Z},$$

and directly implies that

$$\lambda_k^0 \sim ik\pi + o(1), \quad k \in \mathbb{Z} \text{ large enough.}$$

Second case: $e^{t_1+t_3} = \dfrac{b-1}{b+1}$, then $t_1 + t_3 = \ln \dfrac{b-1}{b+1} + ik\pi$, for some $k \in \mathbb{Z}$. As before, we find that the second family of roots of f_0 is of the form

$$\mu_k^0 = \frac{1}{2} \ln \frac{b-1}{b+1} + ik\pi + o(1), \quad k \in \mathbb{Z} \text{ large enough.}$$

Now with the help of Rouché's theorem, we will show that the roots of $\widetilde{f}$ are close to those of f_0. Let us start with the first family. The factor $4e^{-t_1-t_3}(1-b+(1+b)e^{t_1+t_3})$

of f_0 remains bounded and does not vanish in the strip $\frac{1}{4}\Re\ln\frac{b-1}{b+1} \leq \Re(\lambda) \leq 0$. Thus the roots of $\widetilde{f}(\lambda)$ in this strip are also the roots of

$$\widetilde{\widetilde{f}}(\lambda) = \frac{\widetilde{f}(\lambda)e^{t_1+t_3}}{4(1-b+(1+b)e^{t_1+t_3})} = e^{t_1+t_3} - 1 + O(\frac{1}{\lambda}). \qquad (1.3.24)$$

Changing in (1.3.24) the unknown λ by $u = t_1 + t_3$ then (1.3.24) becomes

$$\tilde{\tilde{f}}(u) = e^u - 1 + O(\frac{1}{u}) = \tilde{\tilde{f}}_0(u) + O(\frac{1}{u}).$$

The roots of $\tilde{\tilde{f}}_0$ are $u_k = 2ik\pi, k \in \mathbb{Z}$, and setting $u = u_k + re^{it}, t \in [0, 2\pi]$, we can easily check that there exists a constant $C > 0$ independent of k such that $|e^u - 1| \geq Cr$ for r small enough. This allows to apply Rouché's theorem. Consequently, there exists a subsequence of roots of $\tilde{\tilde{f}}$ which tends to the roots u_k of $\tilde{\tilde{f}}_0$. Equivalently, it means that there exists $N \in \mathbb{N}$ and a subsequence $\{\lambda_k\}_{|k|\geq N}$ of roots of $f(\lambda)$, such that $\lambda_k = \lambda_k^0 + o(1)$ which tends to the roots $\lambda_k^0 = \pm i\frac{2k^2\pi^2}{\sqrt{4k^2\pi^2-b}}$ of f_0. Finally for $|k| \geq N$, λ_k is simple since λ_k^0 is.

The same procedure yields $\mu_k = \frac{1}{2}\ln\frac{b-1}{b+1} + ik\pi + o(1)$.

Our last task is to prove the asymptotic behavior for the first family of eigenvalues near the imaginary axis.

Step 3. From step 2, we can write

$$\lambda_k = ik\pi + \varepsilon_k, \qquad (1.3.25)$$

where $\varepsilon_k = o(1)$. Using (1.3.8)-(1.3.9), we get

$$t_1 + t_3 = 2ik\pi + 2\varepsilon_k + \frac{b}{4(ik\pi)} + o(\varepsilon_k) + O(\frac{1}{k^2}). \qquad (1.3.26)$$

It follows that

$$e^{t_1+t_3} = 1 + 2\varepsilon_k + \frac{b}{4ik\pi} + o(\varepsilon_k) + O(\frac{1}{k^2}). \qquad (1.3.27)$$

Substitute (1.3.27) in (1.3.21), we obtain :

$$f_0(\lambda_k) = -4i\sqrt{b}(4\varepsilon_k + \frac{2b}{4ik\pi} + o(\varepsilon_k) + O(\frac{1}{k^2})). \qquad (1.3.28)$$

Similarly, use (1.3.8)-(1.3.9) in (1.3.22) to find the following asymptotic development

$$\frac{f_1(\lambda_k)}{\lambda_k} = \frac{2b(e^{i\sqrt{b}} - e^{-i\sqrt{b}} + O(\frac{1}{k}))}{ik\pi} + o(\varepsilon_k) = \frac{2b(e^{i\sqrt{b}} - e^{-i\sqrt{b}})}{ik\pi} + O(\frac{1}{k^2}) + o(\varepsilon_k). \tag{1.3.29}$$

Combine (1.3.28)-(1.3.29) and (1.3.20) and using that $\widetilde{f}(\lambda_k) = 0$, we get

$$\widetilde{f}(\lambda_k) = -16i\sqrt{b}\varepsilon_k - \frac{2b\sqrt{b}}{k\pi} + \frac{2b(e^{i\sqrt{b}} - e^{-i\sqrt{b}})}{ik\pi} + O(\frac{1}{k^2}) + o(\varepsilon_k) = 0. \tag{1.3.30}$$

Therefore one has

$$-16i\sqrt{b}\varepsilon_k(1 + o(1)) = \frac{2b\sqrt{b}}{k\pi} - \frac{2b(e^{i\sqrt{b}} - e^{-i\sqrt{b}})}{ik\pi} + O(\frac{1}{k^2}), \tag{1.3.31}$$

and hence

$$\varepsilon_k = \frac{b - 2\sqrt{b}\sin(\sqrt{b})}{8k\pi} i + o(\frac{1}{k}). \tag{1.3.32}$$

This step shows that ε_k is equivalent to a pure imaginary number. Since we are interested in the asymptotic behavior fo the real part of the λ_k, we need to find the next term in the development.

Step 4. From (1.3.32), we can write :

$$\lambda_k = ik\pi + \frac{\alpha}{k} + \frac{\epsilon_k}{k}, \tag{1.3.33}$$

where $\alpha = \dfrac{b - 2\sqrt{b}\sin(\sqrt{b})}{8k\pi} i$ and $\epsilon_k = o(1)$. Using(1.3.8)-(1.3.9) in (1.3.21)-(1.3.23), we obtain

$$\begin{cases} f_0(\lambda_k) = -\dfrac{16i\alpha\sqrt{b}}{k} - \dfrac{2b\sqrt{b}}{k\pi} - \dfrac{16i\sqrt{b}\varepsilon_k}{k} - \dfrac{16i\alpha^2 b\sqrt{b}}{k^2} + \dfrac{ib^3\sqrt{b}}{4k^2\pi^2} - \dfrac{4\alpha b^2\sqrt{b}}{k^2\pi} + O(\dfrac{1}{k^3}) + o(\epsilon_k), \\ \dfrac{f_1(\lambda_k)}{\lambda_k} = \dfrac{4b\sin\sqrt{b}}{k\pi} + O(\dfrac{1}{k^3}) + o(\epsilon_k), \\ \dfrac{f_2(\lambda_k)}{\lambda_k^2} = \dfrac{2ib^2\sqrt{b}(-1+\cos\sqrt{b})}{k^2\pi^2} + O(\dfrac{1}{k^3}) + o(\epsilon_k). \end{cases} \tag{1.3.34}$$

Similarly, as step 3, by substituting (1.3.34) in (1.3.20), we get

$$\epsilon_k = \frac{\beta}{k^2} + o(\frac{1}{k}).$$

with

$$\beta = \frac{-b^2\sin^4(\frac{\sqrt{b}}{2})}{4\pi^2}. \tag{1.3.35}$$

Note that β is negative if $b \neq 4\ell^2\pi^2$, for all $\ell \in \mathbb{N}^*$.

Remark 1.3.2. The asymptotic behavior of the λ_k can be numerically validated. For instance in the case $a = 1, b = 2$, we have calculated numerically some large eigenvalues near the imaginary axis. From (1.3.35) we have in that case $k^2\Re\lambda_k \sim \beta$, with

$$\beta = -\frac{\sin(\frac{1}{\sqrt{2}})^4}{\pi^2} \approx -0.0180461.$$

The table below confirms this behavior.

k	20	40	60	80	100
$k^2\Re\lambda_k$	-0.0180523	-0.0180476	-0.0180468	-0.0180465	-0.0180463

1.3.2 The case $a = 1, b = 4\ell^2\pi^2, \ell \in \mathbb{N}^*$ or $a \neq 1$

An analytic study as in the previous subsection seems difficult even impossible. So, we just provide two numerical examples that exhibit eigenvalues near the imaginary axis. Figure 1.2 represents the eigenvalues in the case $a = 1, b = 4\pi^2$ and Figure 1.3 represents eigenvalues in the case $a = 2, b = 1$.

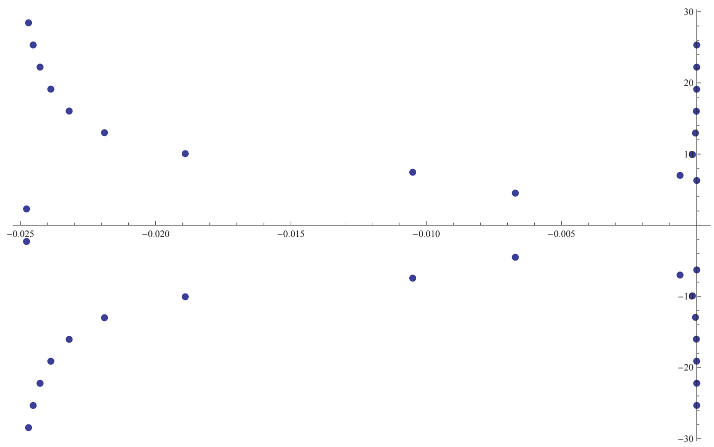

Figure 1.2: Eigenvalues in the case $a = 1$ and $b = 4\pi^2$

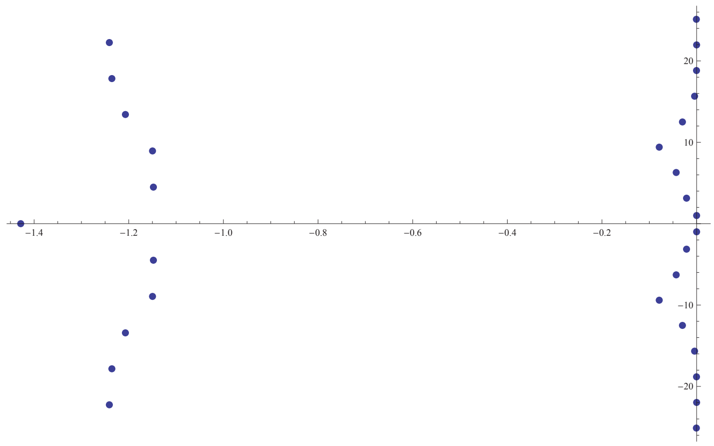

Figure 1.3: Eigenvalues in the case $a = 2$ and $b = 1$

1.4 Polynomial energy decay rate

1.4.1 The case $a = 1$

In this section we state stability results of system (1.1.1)-(3.1.1) under the dissipation law (3.1.2) in the case $a = 1$, $b \neq 1$ and $b \neq 4k^2\pi^2$, for all $k \in \mathbb{Z}^*$. By Lemma 1.3.1, the spectrum of $\mathcal{A}$ is at the left of the imaginary axis, but approaches this axis. Hence, the decay of the energy depends on the asymptotic behavior of the real part of these eigenvalues, since Lemma 1.3.1 shows a behavior like k^{-2}, we can expect a decay rate of the energy of order t^{-1}. This is indeed the case as the next Theorem shows.

Theorem 1.4.1. *Assume that $a = 1$, $b \neq 1$, $b \neq 4\ell^2\pi^2$, $\ell \in \mathbb{Z}^*$, and b satisfies the conditions (C_1) and (C_3). Then there exists $C > 0$ such that for all $U(0) = (u_0, u_1) \in D(\mathcal{A})$, we have*

$$E(t) \leq C \frac{\| U(0) \|^2_{D(\mathcal{A})}}{t}, \quad \forall t > 0. \tag{1.4.1}$$

The proof of this Theorem uses a spectral analysis approaches, namely we show that the set of generalized eigenvectors of $\mathcal{A}$ forms a Riesz basis of $\mathcal{H}$. For that purpose, we need some technical lemmas that we prove first.

Lemma 1.4.2. *Assume that $a = 1$, $b \neq 1$ and $b \neq 4\ell^2\pi^2$, $\ell \in \mathbb{Z}^*$. Then*

1. *There exists a solution $C(\lambda_k)$ of $\widetilde{M}(\lambda_k)C(\lambda_k) = 0$ which has the form :*

$$C(\lambda_k) = C_0 + O(\frac{1}{|\lambda_k|}), \tag{1.4.2}$$

where

$$C_0 = (1, -1, 1, -1).$$

2. *There exists a solution* $C(\mu_k)$ *of* $\widetilde{M}(\mu_k)C(\mu_k) = 0$ *which has the form :*

$$C(\mu_k) = C_0 + O(\frac{1}{|\mu_k|}), \tag{1.4.3}$$

where

$$C_0 = (1, \frac{b-1}{b+1}, -1, -\frac{b-1}{b+1}).$$

Proof Using Lemma 1.3.1, we have

$$\widetilde{M}(\lambda_k) = \widetilde{M_0} + O(\frac{1}{|\lambda_k|}) \tag{1.4.4}$$

where :

$$\widetilde{M_0} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ (-1)^k e^{\frac{i\sqrt{b}}{2}} & (-1)^k e^{\frac{i\sqrt{b}}{2}} & (-1)^k e^{\frac{i\sqrt{b}}{2}} & (-1)^k e^{\frac{i\sqrt{b}}{2}} \\ (-1)^k e^{\frac{i\sqrt{b}}{2}} & -(-1)^k e^{\frac{i\sqrt{b}}{2}} & -(-1)^k e^{\frac{i\sqrt{b}}{2}} & (-1)^k e^{\frac{i\sqrt{b}}{2}} \\ b-1 & -b-1 & -b+1 & b+1 \end{pmatrix}. \tag{1.4.5}$$

Note that $\widetilde{M_0}$ depends only on the parity of k, hence for shortness we have not mentioned its dependence on k. We will continue the proof for k even, the case k odd being fully similar. Note that $\text{rank}(\widetilde{M_0}) = 3$ and that $\widetilde{M_0}C_0 = 0$. From (2.3.5), we can easily show, using Rouché's Theorem that the four eigenvalues of the matrix (1.3.19) are close to the eigenvalues of $\widetilde{M_0}$. Thus, for k large enough, we can find $r > 0$, such that the disk $D(0, r)$ contains only common eigenvalues of $\widetilde{M}(\lambda_k)$ and $\widetilde{M_0}$. Let P_k (resp. P) be the projection matrix on $ker(\widetilde{M}(\lambda_k))$ (resp. $ker(\widetilde{M_0})$), i.e,

$$P_k = \frac{1}{2i\pi}\oint_{\partial D(0,r)} (zI_4 - \widetilde{M}(\lambda_k))^{-1} dz,$$

and,

$$P = \frac{1}{2i\pi}\oint_{\partial D(0,r)} (zI_4 - M_0(\lambda_k))^{-1} dz,$$

where, I_4 is the identity matrix of order 4 . $\forall z \in \partial D(0,r)$, we have

$$(zI_4 - \widetilde{M}(\lambda_k))^{-1} = (zI_4 - M_0(\lambda_k))^{-1}(I_4 - O(\frac{1}{\lambda_k}))^{-1}.$$

For k large enough, using the Neumann series, we get :

$$(zI_4 - \widetilde{M}(\lambda_k))^{-1} = (zI_4 - M_0(\lambda_k))^{-1} + O(\frac{1}{\lambda_k}).$$

Therefore,

$$P_k = P + O(\frac{1}{\lambda_k}).$$

Thus,

$$P_k C_0 = PC_0 + O(\frac{1}{\lambda_k}) = C_0 + O(\frac{1}{\lambda_k}).$$

Setting $C(\lambda_k) = P_k C_0$, then $\widetilde{M}(\lambda_k)C(\lambda_k) = 0$ and satisfies (2.3.25).

We prove similarly (1.4.3) .

Lemma 1.4.3. *Assume that $a = 1$, $b \neq 1$ and $b \neq 4\ell^2\pi^2$, $\ell \in \mathbb{Z}^*$. Then the following results hold.*

1. *Let $\{\phi_k\}_{|k|\geq N}$ be the set of eigenvectors of $\mathcal{A}$ corresponding to $\{\lambda_k\}_{|k|\geq N}$ such that $\|\phi_k\|_{\mathcal{H}} = O(1)$, then we have $\phi_k = \phi_k^0 + O(\frac{1}{k})$, where*

$$\phi_k^0(x) = \frac{1}{ik\pi}(\cos(\frac{\sqrt{b}}{2}x)\sin(k\pi x), ik\pi\cos(\frac{\sqrt{b}}{2}x)\sin(k\pi x), \quad (1.4.6)$$
$$\sqrt{b}\sin(\frac{\sqrt{b}}{2}x)\sin(k\pi x), ik\pi\sqrt{b}\sin(\frac{\sqrt{b}}{2}x)\sin(k\pi x)).$$

2. *Let $\{\psi_k\}_{|k|\geq N}$ be the set of eigenvectors of $\mathcal{A}$ corresponding to $\{\mu_k\}_{|k|\geq N}$ such that $\|\psi_k\|_{\mathcal{H}} = O(1)$, then we have $\psi_k = \psi_k^0 + O(\frac{1}{k})$, where*

$$\psi_k^0(x) = \frac{1}{ik\pi}(\sin(\frac{\sqrt{b}}{2}x)\sinh(\mu_k(x-1)), ik\pi\sin(\frac{\sqrt{b}}{2}x)\sinh(\mu_k(x-1)), \quad (1.4.7)$$
$$-\sqrt{b}\cos(\frac{\sqrt{b}}{2}x)\sinh(\mu_k(x-1)), -ik\pi\sqrt{b}\cos(\frac{\sqrt{b}}{2}x)\sinh(\mu_k(x-1))).$$

Proof. We start by using the finite expansion (1.3.8)-(1.3.17) in (1.3.19) to write

$$\widetilde{M}(\lambda_k) = \widetilde{M}_0 + O(\frac{1}{|\lambda_k|}).$$

To find an eigenvector of $\mathcal{A}$, we substitute $C_0 = (1, -1, 1, -1)$ in (1.3.6) to find :

$$u(x) = e^{t_1 x} - e^{t_2 x} + e^{t_3 x} - e^{t_4 x} + O(\frac{1}{k}).$$

Then, up to a factor, we obtain

$$u(x) = \cos(\frac{\sqrt{b}}{2}x)\sin(k\pi x) + O(\frac{1}{k}). \tag{1.4.8}$$

Then combine (1.4.8) and (3.3.97) to get

$$y(x) = \sqrt{b}\sin(\frac{\sqrt{b}}{2}x)\sin(k\pi x) + O(\frac{1}{k}). \tag{1.4.9}$$

Finally using the first and the third equations of (1.3.1), (1.4.8) and (1.4.9), we have proved (1.4.6) .

We can prove similarly (1.4.7).

For the Riesz basis property of the generalized eigenvectors system of $\mathcal{A}$, we introduce the following auxiliary operator $\mathcal{A}_0$ in $\mathcal{H}$:

$$\mathcal{A}_0 U = (v, u_{xx}, z, y_{xx}), \forall U = (u, v, y, z) \in \mathcal{D}(\mathcal{A}_0) = \mathcal{D}(\mathcal{A}). \tag{1.4.10}$$

Let $\mathcal{H}_j, j = 1, 2$, be the subspaces of $\mathcal{H}$ defined by

$$\mathcal{H}_1 = \{F \in \mathcal{H} | F = (u, v, 0, 0)\},$$

$$\mathcal{H}_2 = \{G \in \mathcal{H} | G = (0, 0, y, z)\}.$$

Lemma 1.4.4. *Assume that $a = 1$, $b \neq 1$ and $b \neq 4\ell^2\pi^2$, $\ell \in \mathbb{Z}^*$. Then*

1. *$\sigma(\mathcal{A}_1) = \{\widetilde{\lambda_k} = ik\pi\}_{k\in\mathbb{Z}^*}$ is the set of eigenvalues of $\mathcal{A}_1 = \mathcal{A}_0|_{\mathcal{H}_1}$, they are simple and the corresponding eigenvectors $\{\widetilde{\phi_k}\}_{k\in\mathbb{Z}^*}$ are given by*

$$\widetilde{\phi_k} = (\widetilde{\lambda_k^{-1}} \sinh(\widetilde{\lambda_k} x), \sinh(\widetilde{\lambda_k x}), 0, 0), \qquad \forall k \in \mathbb{Z}^*,$$

and form an orthogonal basis of $\mathcal{H}_1$.

2. *$\sigma(\mathcal{A}_2) = \{\widetilde{\mu_k} = \frac{1}{2}\ln\frac{b-1}{b+1} + ik\pi, k \in \mathbb{Z}\}$ is the set of eigenvalues of $\mathcal{A}_2 = \mathcal{A}_0|_{\mathcal{H}_2}$, they are simple and the corresponding eigenvectors $\{\widetilde{\psi_k}\}_k, k \in \mathbb{Z}$ are given by*

$$\widetilde{\psi_k} = (0, 0, \widetilde{\mu_k^{-1}} \sinh(\widetilde{\mu_k}(1-x)), \sinh(\widetilde{\mu_k}(1-x))), \qquad \forall k \in \mathbb{Z},$$

and form a Riesz basis of $\mathcal{H}_2$.

3. *The set of eigenvectors of $\mathcal{A}_0$ $\{\widetilde{\phi_k}\}_{k\in\mathbb{Z}^*} \cup \{\widetilde{\psi_k}\}_k \in \mathbb{Z}$ forms a Riesz basis of $\mathcal{H}$.*

Proof. The set $\{\widetilde{\phi_k}\}_{k\in\mathbb{Z}^*}$ is an orthogonal basis of $\mathcal{H}_1$ since $\mathcal{A}_1$ is a skew-adjoint operator. Point (2) is proved in Theorem 3.4.4 of [7]. Point (3) is a direct consequence of the direct decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.

Lemma 1.4.5. *The set of vectors $\{\phi_k^0\}_{k\in\mathbb{Z}^*} \cup \{\psi_k^0\}_{k\in\mathbb{Z}}$ given in Lemma 1.4.3 forms a Riesz basis of $\mathcal{H}$.*

Proof. Let T be the operator defined by :

$$T : \mathcal{H} \;\to\; \mathcal{H} : (u, v, y, z) \;\to\; T(u, v, y, z) = (\widetilde{u}, \widetilde{v}, \widetilde{y}, \widetilde{z}), \tag{1.4.11}$$

where

$$\begin{cases} \widetilde{u}(x) = \cos(\frac{\sqrt{b}}{2}x)u(x) + \sin(\frac{\sqrt{b}}{2}x)y(x), \\ \widetilde{v}(x) = \cos(\frac{\sqrt{b}}{2}x)v(x) + \sin(\frac{\sqrt{b}}{2}x)z(x), \\ \widetilde{y}(x) = \sqrt{b}\sin(\frac{\sqrt{b}}{2}x)u(x) - \sqrt{b}\cos(\frac{\sqrt{b}}{2}x)y(x), \\ \widetilde{z}(x) = \sqrt{b}\sin(\frac{\sqrt{b}}{2}x)v(x) - \sqrt{b}\cos(\frac{\sqrt{b}}{2}x)z(x). \end{cases} \tag{1.4.12}$$

It is easy to check that T is well defined and is a bounded operator from $\mathcal{H}$ to $\mathcal{H}$. Let us now show that it is surjective. For $\widetilde{U} = (\widetilde{u}, \widetilde{v}, \widetilde{y}, \widetilde{z}) \in \mathcal{H}$, we look for $U \in \mathcal{H}$ such that

$$T(U) = \widetilde{U}.$$

Using(1.4.11)-(1.4.12), this is equivalent to

$$\begin{pmatrix} \cos(\frac{\sqrt{b}}{2}x) & \sin(\frac{\sqrt{b}}{2}x) \\ \sqrt{b}\sin(\frac{\sqrt{b}}{2}x) & -\sqrt{b}\cos(\frac{\sqrt{b}}{2}x) \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} \widetilde{u} \\ \widetilde{y} \end{pmatrix},$$

and

$$\begin{pmatrix} \cos(\frac{\sqrt{b}}{2}x) & \sin(\frac{\sqrt{b}}{2}x) \\ \sqrt{b}\sin(\frac{\sqrt{b}}{2}x) & -\sqrt{b}\cos(\frac{\sqrt{b}}{2}x) \end{pmatrix} \begin{pmatrix} v \\ z \end{pmatrix} = \begin{pmatrix} \widetilde{v} \\ \widetilde{z} \end{pmatrix}.$$

Since the determinant of the previous systems is equal to $\sqrt{b}$, we easily find a unique solution (u, v, y, z) that belongs to $\mathcal{H}$. Hence T is an isomorphism from $\mathcal{H}$ into itself. Since the operator T maps $\{\widetilde{\phi}_k\}_{k\in\mathbb{Z}^*} \cup \{\widetilde{\psi}_k\}_{k\in\mathbb{Z}}$ into $\{\phi_k^0\}_{k\in\mathbb{Z}^*} \cup \{\psi_k^0\}_{k\in\mathbb{Z}}$, and using Lemma 1.4.4, we deduce that $\{\phi_k^0\}_{k\in\mathbb{Z}^*} \cup \{\psi_k^0\}_{k\in\mathbb{Z}}$ forms a Riesz basis of $\mathcal{H}$. Finally, if $a = 1$, $b \neq 1$ and $b \neq 4\ell^2\pi^2$, $\ell \in \mathbb{Z}^*$., we will prove that the set of eigenvectors of $\mathcal{A}$ forms a Riesz basis of $\mathcal{H}$. For this aim, we use Theorem 6.3 of [24] which is a new form of Bari's Theorem (see Theorem 2.3 of Chapter VI in [23], see also Theorem 1.2.8 of [6]), recalled below.

Theorem 1.4.6. *Let $\mathcal{A}$ be a densely defined operator in a Hilbert space $\mathcal{H}$ with a compact resolvent. Let $\{\varphi_n\}_1^\infty$ be a Riesz basis of $\mathcal{H}$. If there are an integer $N \geq 0$ and a sequence of generalized eigenvectors $\{\psi_n\}_{N+1}^\infty$ of $\mathcal{A}$ such that*

$$\sum_{N+1}^{\infty} \|\varphi_n - \psi_n\|^2 < \infty,$$

then the set of generalized eigenvectors (or root vectors) of $\mathcal{A}$ forms a Riesz basis of $\mathcal{H}$.

Theorem 1.4.7. *Assume that $a = 1$, $b \neq 1$ and $b \neq 4\ell^2\pi^2$, $\ell \in \mathbb{Z}^*$. Then the set of generalized eigenvectors of $\mathcal{A}$ forms a Riesz basis of the energy space $\mathcal{H}$.*

Proof. From Lemma 1.4.3, we have

$$\|\phi_k - \phi_k^0\|_{\mathcal{H}} = O(\frac{1}{k}), \quad \text{and} \quad \|\psi_k - \psi_k^0\|_{\mathcal{H}} = O(\frac{1}{k}).$$

We conclude the desired aim by Theorem 1.4.6.

Corollary 1.4.8. *Assume that $a = 1$, $b \neq 1$ and $b \neq 4\ell^2\pi^2$, $\ell \in \mathbb{Z}^*$.. Then there exist $N \in \mathbb{N}$ and J a set of finite index such that the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ admits the splitting*

$$\sigma(\mathcal{A}) = \{\lambda_k\}_{|k| \geq N} \cup \{\mu_k\}_{|k| \geq N} \cup \{\eta_j\}_{j \in J}. \tag{1.4.13}$$

If m_j denotes the multiplicity of η_j for every $j \in J$, then we can denote by

$$\{\{\varphi_{j,i}\}_{i=0}^{m_j - 1}\}_{j \in J} \cup \{\phi_k\}_{|k| \geq N} \cup \{\psi_k\}_{|k| \geq N} \tag{1.4.14}$$

the set of generalized eigenvectors of $\mathcal{A}$. From the previous considerations, they form a Riesz basis of $\mathcal{H}$.

We are now ready to prove the main result of this subsection.

Proof of Theorem 1.4.1. By writing the solution $U(t)$ in the Riesz basis defined in (1.4.14), using Corollary 1.4.8 (by the assumptions on b we here have $\Re\eta_j < 0$ for all $j \in J$), Lemma 1.3.1 and the fact that $E(t) = \frac{1}{2}\|U(t)\|_{\mathcal{H}}^2$, we get (1.4.1).

Remark 1.4.9. If $a = 1$ and $b = 4\ell^2\pi^2$, for some $\ell \in \mathbb{Z}^*$, the asymptotic behavior of the λ_k stated in Lemma 1.3.1 fails since $\beta = 0$. In this case we have tried to find a more precise asymptotic behavior of the λ_k, by a formal antsatz (and the help of an automated computation system) we found

$$\lambda_k = ik\pi + \sum_{j=0}^{4} \frac{\alpha_{2j+1}}{k^{2j+1}} + \frac{\alpha_{10}}{k^{10}} + o(\frac{1}{k^{10}}), k \in \mathbb{Z}^*, |k| \geq N,$$

with $\alpha_{2j+1} \in i\mathbb{R}$ and $\alpha_{10} < 0$. We unfortunately were not able to prove this expansion, but we conjecture a decay rate $t^{-1/5}$ of the energy for initial data in $D(\mathcal{A})$.

1.4.2 The case $a \neq 1$.

Under the non equal speed propagation ($a \neq 1$) and if $\sqrt{a}$ is a rational number, we obtain the polynomial stability of system (1.1.1)-(1.1.5) :

Theorem 1.4.10. *(Polynomial decay rate)*
Assume that $a \neq 1$ and b satisfies the conditions (C_1), (C_2) and (C_3). Moreover, assume that $\sqrt{a} \in \mathbb{Q}$, then there exists a positive constant $c > 0$ such that for all initial $(u_0, u_1, y_0, y_1) \in D(\mathcal{A})$ the energy of the system (1.1.1)-(1.1.5) satisfies the following decay rate :

$$E(t) \leq \frac{c}{\sqrt[3]{t}} \|E(0)\|^2_{D(\mathcal{A})}, \quad \forall t > 0. \tag{1.4.15}$$

Proof. Using theorem 2.4 of [16] (see also [28]) a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in a Hilbert space $\mathcal{H}$ satisfies (1.4.15) if

$$i\mathbb{R} \subset \rho(\mathcal{A}) \qquad \text{(H1)}$$

and

$$\sup_{\lambda \in \mathbb{R}} \frac{1}{|\lambda|^6} \|(i\lambda I - \mathcal{A})^{-1}\| < +\infty \qquad \text{(H2)}$$

hold. As condition (H1) was already checked, we now prove that condition (H2) holds, using again an argument of contradiction.

Suppose that (H2) is false. Then there exists a sequence $\lambda_n \in \mathbb{R}$ and a sequence $(u^n, v^n, y^n, z^n) \in D(\mathcal{A})$, such that

$$|\lambda_n| \to +\infty, \qquad \|(u^n, v^n, y^n, z^n)\|_{\mathcal{H}} = 1, \tag{1.4.16}$$

$$\lambda_n^6 (i\lambda_n I - \mathcal{A})(u^n, v^n, y^n, z^n) = (f_1^n, g_1^n, f_2^n, g_2^n) \to 0 \qquad \text{in } \mathcal{H}. \tag{1.4.17}$$

Taking the inner product of (1.4.17) with $U^n = (u^n, v^n, y^n, z^n)$ and using (**??**), we get

$$\lambda_n^6 \Re (i\lambda_n - \mathcal{A}) U^n, U^n)_{\mathcal{H}} = \lambda_n^6 |z^n(0)|^2 = o(1). \tag{1.4.18}$$

For simplicity, from now on, we drop now the index n.

The first and third equations of (1.4.17) being equivalent to

$$v = i\lambda u - \frac{f_1}{\lambda^6} \qquad \text{and} \quad z = i\lambda y - \frac{f_2}{\lambda^6}, \tag{1.4.19}$$

and by substitution in the second and fourth equations, we arrive at the system

$$\begin{cases} \lambda^2 u + u_{xx} + y_x = f \\ \lambda^2 y + a y_{xx} - by - bu_x = g \\ u(0) = u(1) = y(1) = 0, \end{cases} \tag{1.4.20}$$

with

$$f = -\frac{g_1 + i\lambda f_1}{\lambda^6}, \quad g = -\frac{g_2 + i\lambda f_2}{\lambda^6}.$$

Our aim is to prove that

$$u_x(0) = o(1). \tag{1.4.21}$$

If this is not true, up to subsequence we can assume $u_x^n(0) = 1, \forall n \in \mathbb{N}$.

Now the system (1.4.20) can be written as

$$U' = BU+F, \qquad \text{where } U = \begin{pmatrix} u \\ u_x \\ y \\ y_x \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\lambda^2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & \widetilde{b} & \widetilde{b} - \widetilde{a}\lambda^2 & 0 \end{pmatrix} \text{ and } F = \begin{pmatrix} 0 \\ f \\ 0 \\ \widetilde{a}g \end{pmatrix},$$

where $\tilde{a} = \dfrac{1}{a}$ and $\tilde{b} = \dfrac{b}{a}$. For shortness later on we replace $\tilde{a}$ by a and $\tilde{b}$ by b.
A straightforward computation show that the eigenvalues μ of the matrix B are the roots of the following equation

$$x^4 + \lambda^2(a+1)x^2 + a\lambda^4 - b\lambda^2 = 0. \tag{1.4.22}$$

The discriminant is:

$$\Delta = (a+1)^2\lambda^4 - 4a\lambda^4 + 4b\lambda^2 = (a-1)^2\lambda^4 + 4b\lambda^2 > 0. \tag{1.4.23}$$

Thus (1.4.22) has only pure imaginary solutions when λ is large enough. Applying the variation of constants formula, we obtain

$$U(x) = U_0(x) + \int_0^x W(x-s)F(s)ds, \tag{1.4.24}$$

where W is the solution of the homogeneous equation

$$\frac{dW}{dx} = BW, \qquad W(0) = I \tag{1.4.25}$$

and U_0 is the solution of the homogeneous equation

$$\frac{dU_0}{dx} = BU_0, \qquad U_0(0) = (0, 1, y(0), y_x(0))^\top. \tag{1.4.26}$$

To obtain an explicit expression of (1.4.24), we consider the initial value problem

$$\begin{cases} \lambda^2 u + u_{xx} + y_x = 0, \\ \lambda^2 y + a y_{xx} - by = 0, \\ u(0) = c_1; u_x(0) = c_2; y(0) = c_3; y_x(0) = c_4. \end{cases} \tag{1.4.27}$$

Then a straightforward computation gives that :

$$\begin{cases} u = Ae^{\mu_1 x} + Be^{-\mu_1 x} + Ce^{\mu_2 x} + De^{-\mu_2 x}, \\ \\ y = -A\Big(\frac{\lambda^2}{\mu_1} + \mu_1\Big)e^{\mu_1 x} + B\Big(\frac{\lambda^2}{\mu_1} + \mu_1\Big)e^{-\mu_1 x} - C\Big(\frac{\lambda^2}{\mu_2} + \mu_2\Big)e^{\mu_2 x} + D\Big(\frac{\lambda^2}{\mu_2} + \mu_2\Big)e^{-\mu_2 x}, \end{cases} \tag{1.4.28}$$

where

$$\begin{cases} A + B + C + D = c_1 \\ \mu_1 A - \mu_1 B + \mu_2 C - \mu_2 D = c_2 \\ -\dfrac{\mu_1^2 + \lambda^2}{\mu_1}A + \dfrac{\mu_1^2 + \lambda^2}{\mu_1}B - \dfrac{\mu_2^2 + \lambda^2}{\mu_2}C + \dfrac{\mu_2^2 + \lambda^2}{\mu_2}D = c_3 \\ -\Big(\mu_1^2 + \lambda^2\Big)A - \Big(\mu_1^2 + \lambda^2\Big)B - \Big(\mu_2^2 + \lambda^2\Big)C - \Big(\mu_2^2 + \lambda^2\Big)D = c_4, \end{cases} \tag{1.4.29}$$

and $\pm\mu_i, i = 1, 2$ are the roots of (1.4.22). Using Maple, we find that

$$\begin{cases} A = -\frac{1}{2}\dfrac{\mu_2^2 + \lambda^2}{\mu_1^2 - \mu_2^2}c_1 + \frac{1}{2}\dfrac{(\mu_2^2 + \lambda^2)\mu_1 c_2}{(\mu_1^2 - \mu_2^2)\lambda^2} + \frac{1}{2}\dfrac{\mu_2^2\mu_1 c_3}{(\mu_1^2 - \mu_2^2)\lambda^2} - \frac{1}{2}\dfrac{c_4}{\mu_1^2 - \mu_2^2}, \\ \\ B = -\frac{1}{2}\dfrac{\mu_2^2 + \lambda^2}{\mu_1^2 - \mu_2^2}c_1 - \frac{1}{2}\dfrac{(\mu_2^2 + \lambda^2)\mu_1 c_2}{(\mu_1^2 - \mu_2^2)\lambda^2} - \frac{1}{2}\dfrac{\mu_2^2\mu_1 c_3}{(\mu_1^2 - \mu_2^2)\lambda^2} - \frac{1}{2}\dfrac{c_4}{\mu_1^2 - \mu_2^2}, \\ \\ C = \frac{1}{2}\dfrac{\mu_1^2 + \lambda^2}{\mu_1^2 - \mu_2^2}c_1 - \frac{1}{2}\dfrac{(\mu_1^2 + \lambda^2)\mu_2 c_2}{(\mu_1^2 - \mu_2^2)\lambda^2} - \frac{1}{2}\dfrac{\mu_1^2\mu_2 c_3}{(\mu_1^2 - \mu_2^2)\lambda^2} + \frac{1}{2}\dfrac{c_4}{\mu_1^2 - \mu_2^2}, \\ \\ D = \frac{1}{2}\dfrac{\mu_1^2 + \lambda^2}{\mu_1^2 - \mu_2^2}c_1 + \frac{1}{2}\dfrac{(\mu_1^2 + \lambda^2)\mu_2 c_2}{(\mu_1^2 - \mu_2^2)\lambda^2} + \frac{1}{2}\dfrac{\mu_1^2\mu_2 c_3}{(\mu_1^2 - \mu_2^2)\lambda^2} + \frac{1}{2}\dfrac{c_4}{\mu_1^2 - \mu_2^2}. \end{cases} \tag{1.4.30}$$

Setting (c_1, c_2, c_3, c_4) to be the unit vectors e_i for $i = 1, ..., 4$, we get

$$\begin{cases} u_1 = -\dfrac{\mu_2^2 + \lambda^2}{\mu_1^2 - \mu_2^2}\cosh(\mu_1 x) + \dfrac{\mu_1^2 + \lambda^2}{\mu_1^2 - \mu_2^2}\cosh(\mu_2 x), \\ \\ y_1 = \dfrac{(\mu_1^2 + \lambda^2)(\mu_2^2 + \lambda^2)}{\mu_1(\mu_1^2 - \mu_2^2)}\sinh(\mu_1 x) - \dfrac{(\mu_1^2 + \lambda^2)(\mu_2^2 + \lambda^2)}{\mu_2(\mu_1^2 - \mu_2^2)}\sinh(\mu_2 x), \end{cases} \tag{1.4.31}$$

$$\begin{cases} u_2 = \dfrac{(\mu_2^2+\lambda^2)\mu_1}{\lambda^2(\mu_1^2-\mu_2^2)}\sinh(\mu_1 x) - \dfrac{(\mu_1^2+\lambda^2)\mu_2}{\lambda^2(\mu_1^2-\mu_2^2)}\sinh(\mu_2 x), \\ \\ y_2 = -\dfrac{(\mu_1^2+\lambda^2)(\mu_2^2+\lambda^2)}{\lambda^2(\mu_1^2-\mu_2^2)}\cosh(\mu_1 x) + \dfrac{(\mu_1^2+\lambda^2)(\mu_2^2+\lambda^2)}{\lambda^2(\mu_1^2-\mu_2^2)}\cosh(\mu_2 x), \end{cases} \tag{1.4.32}$$

$$\begin{cases} u_3 = \dfrac{\mu_2^2\mu_1}{\lambda^2(\mu_1^2-\mu_2^2)}\sinh(\mu_1 x) - \dfrac{\mu_1^2\mu_2}{\lambda^2(\mu_1^2-\mu_2^2)}\sinh(\mu_2 x), \\ \\ y_3 = -\dfrac{(\mu_1^2+\lambda^2)\mu_2^2}{\lambda^2(\mu_1^2-\mu_2^2)}\cosh(\mu_1 x) + \dfrac{(\mu_2^2+\lambda^2)\mu_1^2}{\lambda^2(\mu_1^2-\mu_2^2)}\cosh(\mu_2 x), \end{cases} \tag{1.4.33}$$

$$\begin{cases} u_4 = -\dfrac{1}{\mu_1^2-\mu_2^2}\cosh(\mu_1 x) + \dfrac{1}{\mu_1^2-\mu_2^2}\cosh(\mu_2 x), \\ \\ y_4 = \dfrac{\mu_1^2+\lambda^2}{\mu_1(\mu_1^2-\mu_2^2)}\sinh(\mu_1 x) - \dfrac{\mu_2^2+\lambda^2}{\mu_2(\mu_1^2-\mu_2^2)}\sinh(\mu_2 x). \end{cases} \tag{1.4.34}$$

Expansion of μ for $a > 1$. From (1.4.23), we have

$$2\mu_{1,2}^2 = -(a+1)^2\lambda^2 \pm \sqrt{(a-1)^2\lambda^4 + 4b\lambda^2}. \tag{1.4.35}$$

It follows that

$$\mu_1 = i\lambda - i\frac{b}{2(a-1)\lambda} + i\frac{(5-a)b^2}{8(a-1)^3\lambda^3} + i\frac{(6a-a^2-21)b^3}{16(a-1)^5\lambda^5} + O(\frac{1}{\lambda^7}), \tag{1.4.36}$$

$$\mu_2 = i\sqrt{a}\lambda + i\frac{b}{2\sqrt{a}(a-1)\lambda} - i\frac{(5a-1)b^2}{8\sqrt{a}a(a-1)^3\lambda^3} + \frac{(21a^2-6a+1)b^3}{16a^2\sqrt{a}(a-1)^5\lambda^5} + O(\frac{1}{\lambda^7}), \tag{1.4.37}$$

$$\mu_1^2 - \mu_2^2 = (a-1)\lambda^2 + \frac{2b}{a-1} - \frac{2b^2}{(a-1)^3\lambda^2} + \frac{4b^3}{(a-1)^5\lambda^4} + O(\frac{1}{\lambda^6}), \tag{1.4.38}$$

$$\mu_1^2 + \lambda^2 = \frac{b}{a-1} - \frac{b^2}{(a-1)^3\lambda^2} + \frac{2b^3}{(a-1)^5\lambda^4} + O(\frac{1}{\lambda^6}), \tag{1.4.39}$$

$$\mu_2^2 + \lambda^2 = (1-a)\lambda^2 - \frac{b}{a-1} + \frac{b^2}{(a-1)^3\lambda^2} - \frac{2b^3}{(a-1)^5\lambda^4} + O(\frac{1}{\lambda^6}). \tag{1.4.40}$$

Therefore

$$\begin{cases} u_1(x) = \cosh(\mu_1 x) + O(\frac{1}{\lambda^2}), \\ y_1(x) = i\frac{b}{\lambda(a-1)}\sinh(\mu_1 x) - i\frac{\tilde{b}}{\sqrt{a}(a-1)\lambda}\sinh(\mu_2 x) + O(\frac{1}{\lambda^3}), \\ u_2(x) = \frac{1}{i\lambda}\sinh(\mu_1 x) + O(\frac{1}{\lambda^3}), \\ y_2(x) = (\frac{b}{(a-1)\lambda^2} - \frac{2b^2}{(a-1)^3\lambda^4})(\cosh(\mu_1 x) - \cosh(\mu_2 x)) + O(\frac{1}{\lambda^6}), \\ u_3(x) = -\frac{-ia}{(a-1)\lambda}\sinh(\mu_1 x) + \frac{i\sqrt{a}}{(a-1)\lambda}\sinh(\mu_2 x) + O(\frac{1}{\lambda^3}), \\ y_3(x) = \frac{ab}{(a-1)\lambda^2}\cosh(\mu_1 x) + \cosh(\mu_2 x) + O(\frac{1}{\lambda^4}), \\ u_4(x) = \frac{-1}{(a-1)\lambda^2}\cosh(\mu_1 x) + \frac{1}{(a-1)\lambda^2}\cosh(\mu_2 x) + O(\frac{1}{\lambda^2}), \\ y_4(x) = \frac{b}{i(a-1)^2\lambda^3}\sinh(\mu_1 x) + \frac{1}{i\sqrt{a}\lambda}\sinh(\mu_2 x) + O(\frac{1}{\lambda^5}). \end{cases} \tag{1.4.41}$$

Noticing that

$$W = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 \\ u_1' & u_2' & u_3' & u_4' \\ y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \end{pmatrix} \tag{1.4.42}$$

we deduce from (1.4.24) and (1.4.42) that

$$\begin{cases} u = u_2 + u_3 y(0) + u_4 y'(0) + \int_0^x (f(s)u_2(x-s) + ag(s)u_4(x-s))ds, \\ \\ y = y_2 + y_3 y(0) + y_4 y'(0) + \int_0^x (f(s)y_2(x-s) + ag(s)y_4(x-s))ds. \end{cases} \tag{1.4.43}$$

Using (1.4.18) we get $z(0) = o(\frac{1}{\lambda^3})$. Consequently we deduce that $y_x(0) = o(\frac{1}{\lambda^3})$ and $y(0) = o(\frac{1}{\lambda^4})$. Then combining with (1.4.41) and (1.4.43), we get:

$$\begin{cases} u(x) = u_2(x) + \frac{o(1)}{\lambda^5}, \\ \\ y(x) = y_2(x) + \frac{o(1)}{\lambda^4}. \end{cases} \tag{1.4.44}$$

Now applying the boundary conditions $u(1) = y(1) = 0$, the expression of (1.4.44) and (1.4.41) lead to

$$\begin{cases} \sinh(\mu_1) = \dfrac{O(1)}{\lambda^2}, \\ \cosh(\mu_2) = \cosh(\mu_1) + \dfrac{o(1)}{\lambda^2}. \end{cases} \tag{1.4.45}$$

Then using (1.4.36)-(1.4.37), it follows from (1.4.45) that there exist $m, k \in \mathbb{Z}$ with the same parity such that

$$\begin{cases} \lambda - \dfrac{b}{2(a-1)\lambda} = m\pi + \dfrac{O(1)}{\lambda^2}, \\ \sqrt{a}\lambda + \dfrac{b}{2\sqrt{a}(a-1)\lambda} = k\pi + \dfrac{o(1)}{\lambda}. \end{cases} \tag{1.4.46}$$

Since $m \sim k \sim \lambda$, (1.4.46) can be written as

$$\begin{cases} \lambda^2 = m^2\pi^2 + \dfrac{b\pi}{(a-1)} + \dfrac{O(1)}{\lambda}, \\ a\lambda^2 = k^2\pi^2 - \dfrac{b\pi}{\sqrt{a}(a-1)} + o(1). \end{cases} \tag{1.4.47}$$

Finally we obtain

$$(am^2 - k^2) = -\frac{b(a\sqrt{a}+1)}{\sqrt{a}(a-1)\pi} + o(1). \tag{1.4.48}$$

Let us set

$$c = -\frac{b(a+1)}{(a-1)\pi^2}.$$

Since we have assumed that $a = \dfrac{p^2}{q^2}$ for some $p, q \in \mathbb{N}$, we deduce

$$\frac{pm - qk}{q^2} = \frac{c}{pm+qk} + \frac{o(1)}{pm+qk}. \tag{1.4.49}$$

i) If $pm - qk = 0$ for an infinity number of pairs (m, k), then $c = o(1)$ and this a contradiction.

ii) Else $pm - kn \neq 0$ for λ large enough and then

$$\frac{1}{q^2} \leq \left|\frac{c}{pm+qk}\right| + \left|\frac{o(1)}{pm+qk}\right|,$$

which cannot be true.

The remainder of the proof is based on the classical multiplier method. For the sake of completeness, here we give a sketch of the procedure. Multiplying the first equation of (1.4.20) by $2h(x)\overline{u_x}$ and the second one by $2h(x)\overline{y_x}$ and integrating by parts to get

$$\int_0^1 h'(x)|\lambda u|^2dx+\int_0^1 h'(x)|u_x|^2dx-2\Re\left(\int_0^1 h(x)y_x\overline{u_x}dx\right)+h(0)|u_x(0)|^2-h(1)|u_x(1)|^2 = o(1), \tag{1.4.50}$$

$$\int_0^1 h'(x)|\lambda y|^2dx+a\int_0^1 h'(x)|y_x|^2dx+2\Re\left(\int_0^1 bh(x)u_x\overline{y_x}dx\right)-ah(1)|y_x(1)|^2 = o(1). \tag{1.4.51}$$

For $h(x) = 1$ in (1.4.50) and (1.4.51) we can deduce that

$$b|u_x(0)|^2 - b|u_x(1)|^2 - a|y_x(1)|^2 = o(1). \tag{1.4.52}$$

Inserting (1.4.21) into (1.4.52), we obtain

$$u_x(1) = o(1), \tag{1.4.53}$$

$$y_x(1) = o(1). \tag{1.4.54}$$

Multiply (1.4.50) by b and using (1.4.53)-(1.4.54), we get

$$b\int_0^1 |\lambda u|^2dx + b\int_0^1 |u_x|^2dx + \int_0^1 |\lambda y|^2dx + a\int_0^1 |y_x|^2dx = o(1). \tag{1.4.55}$$

Hence, with (1.4.55) we obtain a contradiction with (1.4.16). The proof is thus complete.

Conclusion

We have studied the indirect boundary stabilization of the Timoshenko system with only one dissipation law. If the wave speeds are equal ($a = 1$) and if b

is outside a discrete set of exceptional values, using a spectral analysis, we have proved a non uniform stability, the same non uniform stability is expected in the other cases but is only checked numerically in some examples. If the wave speeds are equal ($a = 1$) and if b is outside a discrete set of exceptional values, using a Riesz basis method, we prove the optimal polynomial energy decay rate in $\frac{1}{t}$. If $\sqrt{a}$ is a rational number and if b is outside another discrete set of exceptional values, using a frequency domain approach, we prove some polynomial energy decay rate. The remaining cases could be analyzed in the same way with a slower polynomial decay rate. This will be investigated in the future.

Chapter 2

Optimal energy decay rate of Rayleigh beam equation with only one boundary control force

2.1 introduction

We consider a clamped Rayleigh beam equation. The system is governed by the following partial differential equations:

$$y_{tt} - \gamma y_{xxtt} + y_{xxxx} = 0, \quad 0 < x < 1, \ t > 0, \tag{2.1.1}$$

$$y(0,t) = y_x(0,t) = 0, \quad t > 0, \tag{2.1.2}$$

$$y_{xx}(1,t) + \alpha y_{xt}(1,t) = 0, \quad t > 0, \tag{2.1.3}$$

$$y_{xxx}(1,t) - \gamma y_{xtt}(1,t) - \beta y_t(1,t) = 0, \quad t > 0 \tag{2.1.4}$$

where $\gamma > 0$ is the coefficient of moment of inertia, $\beta > 0$ is the coefficient of the control force and where $\alpha \geq 0$ is the coefficient of the control moment (in this paper we consider the case of only one control force *i.e.* $\alpha = 0$ and $\beta > 0$). For more details concerning the modeling of the system, we refer to Russell [45].

If $\gamma = 0$ the Rayleigh beam equation simplifies to the Euler-Bernoulli beam equation. But, in the case of one control force $\alpha = 0$ and $\beta > 0$, the nature of the stabilization of the Rayleigh beam equation is different from that of Euler-Bernoulli beam equation. In fact, Chen *et al.* [18], [19] (see also [30]) proved the uniform stability of Euler-Bernoulli beam equation while Rao in [39], proved the strong but nonuniform stability of Rayleigh beam equation if and only if the inertia coefficient γ is large enough. However, in this case of only one control force, no rate of decay has been discussed. We refer the reader to the references [18], [19], [40], [41] [31] and [5] for the Euler-Bernoulli beam equation with different kinds of damping mechanisms.

Now, concerning the Rayleigh beam equation ($\gamma > 0$), different types of dampings have been introduced to the Rayleigh beam equation and several stability results have been obtained. Rao [42] studied the stabilization of Rayleigh beam equation subject to a positive internal viscous damping. Using a constructive approximation, he established the optimal exponential energy decay rate. Wehbe in [50], considered the Rayleigh beam equation with two dynamical boundary feedbacks. First, using a compact perturbation method, he proved that the Rayleigh beam equation is not uniformly stable *i.e.* the non-exponential energy decay rate. Next, using a spectral method, he established the optimal polynomial energy decay rate for smooth initial data. In [27], Lagnese studied the stabilization of system (2.1.1)-(2.1.4) with two boundary control (the case $\alpha > 0$ and $\beta > 0$). He proved that the energy decays exponentially to zero for all initial data. Rao [39] extended the results of [27] to the case of one boundary feedback (the case $\alpha > 0$, $\beta = 0$ or $\alpha = 0$, $\beta > 0$). In the case of one control moment (the case $\alpha > 0$ and $\beta = 0$), using a compact perturbation theory due to Gibson [22], he established an exponential stability of system (2.1.1)-(2.1.4). Moreover, in the case one control force ($\alpha = 0$, $\beta > 0$), he first, proved the lack of exponential stability of system (2.1.1)-(2.1.4). Next, he proved that the Rayleigh beam equation can be strongly stabilized by only one control force if and only if the inertia coefficient γ is large enough but no decay rate has been discussed.

Nevertheless, in the case one control force ($\alpha = 0$, $\beta > 0$) the energy decay rate and it's optimality appears to be an open problem. Then, in this paper, we consider the Rayleigh beam equation with only one boundary control. Using an explicit approximation, we give the asymptotic asymptotic expansion of eigenvalues and eigenfunctions of the undamped system corresponding to system (2.1.1)-(2.1.4). This yields, on the first hand, to establish an observability inequality of solution of the undamped system and on the second hand, to verify the boundedness property of the transfer function associated to the undamped problem. Then, using a methodology introduced in [4], we establish a polynomial energy decay rate of type $1/t$ for smooth initial data. Finally, from an explicit approximation, we give the real part of the asymptotic expansion of the eigenvalues of system (2.1.1)-(2.1.4). This combining with a frequency domain approach, yields to prove that the obtained energy decay rate is optimal.

We now outline briefly the content of this paper. In section 2, in a convenable Hilbert space, we formulate system (2.1.1)-(2.1.4) into an evolution equation and we recall the well-posedness property of the problem by the semigroup approach (see [38], [39]). In section 3, we propose an explicit approximation of the characteristic determinant of the undamped system corresponding to system (2.1.1)-(2.1.4) and we obtain an asymptotic expansion of eigenvalues and eigenfunctions of the corresponding operator. In section 4, we establish a polynomial energy decay rate for smooth initial data. In section 5, we prove that the obtained energy decay rate is optimal.

Here and after we consider system (2.1.1)-(2.1.4) with $\alpha = 0$ and $\beta > 0$.

2.2 Well-posedness and strong stability of the problem

The aim of this section is to study existence, uniqueness and asymptotic behavior of the solution of system (2.1.1)-(2.1.4). We first introduce the following spaces

$$V = \{y \in H^1(0,1) : y(0) = 0\}, \;\; \|y\|_V^2 = \int_0^1 (|y|^2 + \gamma |y_x|^2)dx,$$

$$W = \{y \in H^2(0,1) : y(0) = y_x(0) = 0\}, \;\; \|y\|_W^2 = \int_0^1 |y_{xx}|^2 dx$$

and the energy space $\mathcal{H}$ as $\mathcal{H} = W \times V$ which is endowed with the usual inner product

$$\big((y,z),(\widetilde{y},\widetilde{z})\big)_{\mathcal{H}} = (y,\widetilde{y})_W + (z,\widetilde{z})_V, \;\; \forall (y,z),\, (\widetilde{y},\widetilde{z}) \in \mathcal{H}.$$

Identify $L^2(0,1)$ with its dual so that we have the following continuous embedding

$$W \subset V \subset L^2(0,1) \subset V' \subset W'. \tag{2.2.1}$$

Let y a smooth solution of system (2.1.1)-(2.1.4). Then multiplying (2.1.1) by a function $\varphi \in W$ and integrating by parts, we get

$$\int_0^1 (y_{tt}\overline{\varphi} + \gamma y_{xtt}\overline{\varphi_x})dx + \int_0^1 y_{xx}\overline{\varphi_{xx}}dx + \beta y_t(1)\overline{\varphi(1)} = 0. \tag{2.2.2}$$

Accordingly, we define the linear operators $\widetilde{A} \in L(W,W')$, $\widetilde{B} \in L(V,V')$, $C \in L(V,V')$, by

$$< \widetilde{A}y, \varphi >_{W' \times W} = (y,\varphi)_W, \;\; \forall y,\, \varphi \in W, \tag{2.2.3}$$

$$< \widetilde{B}y, \varphi >_{V' \times V} = y(1)\varphi(1), \;\; \forall y,\, \varphi \in V, \tag{2.2.4}$$

$$< Cy, \varphi >_{V' \times V} = (y,\varphi)_V, \;\; \forall y,\, \varphi \in V. \tag{2.2.5}$$

By means of Lax-Milgram's theorem (see [17]), we see that $\widetilde{A}$, C is the canonical isomorphism from W onto W' and from V onto V' respectively. On the other hand, using the usual traces theorems, we check easily that $\widetilde{B}$ is continuous operator for the corresponding topology.

Assume that $\widetilde{A}y \in V'$, then we can formulate the variational equation (2.2.2) into the following form

$$y_{tt} + C^{-1}\widetilde{A}y + \beta C^{-1}\widetilde{B}y_t = 0, \qquad \text{in} \quad V. \tag{2.2.6}$$

Next we introduce the linear unbounded operator $\mathcal{A}_0$ by

$$D(\mathcal{A}_0) = \big\{(y, z) \in \mathcal{H} \ : z \in W \text{ and } \widetilde{A}y \in V'\big\}, \tag{2.2.7}$$

$$\mathcal{A}_0 u = (z, -C^{-1}\widetilde{A}y), \quad \forall u = (y, z) \in D(\mathcal{A}_0) \tag{2.2.8}$$

and the linear bounded operator $\mathcal{B}_\beta$ as follows

$$\mathcal{B}_\beta u = (0, -\beta C^{-1}\widetilde{B}z), \quad \forall u = (y, z) \in \mathcal{H}. \tag{2.2.9}$$

Then, denoting by $u = (y, y_t)$ the state of system (2.2.6), we can formulate (2.2.6) into an evolution equation

$$\begin{cases} u_t = (\mathcal{A}_0 + \mathcal{B}_\beta)u, \\ u(0) = u_0 \in \mathcal{H}. \end{cases} \tag{2.2.10}$$

It is easy to prove that $\mathcal{A}_0$ is maximal dissipative and $\mathcal{B}_\beta$ is dissipative in the energy space $\mathcal{H}$, therefore $\mathcal{A} = \mathcal{A}_0 + \mathcal{B}_\beta,\ D(\mathcal{A}) = D(\mathcal{A}_0)$, generates a C_0-semigroup $e^{t\mathcal{A}}$ of contractions on the energy space $\mathcal{H}$ following Hille-Yosida's theorem (see [38]). Then we have the following results concerning existence and uniqueness of solution of the problem (2.2.10)

Theorem 2.2.1. *For any initial data $u_0 \in \mathcal{H}$, the problem (2.2.10) has a unique weak solution $u(t) \in \mathcal{C}^0([0, \infty), \mathcal{H})$. Moreover, if $u_0 \in D(\mathcal{A})$, then the problem (2.2.10) has a strong solution $u(t) \in \mathcal{C}^1([0, \infty), \mathcal{H}) \cap \mathcal{C}^0([0, \infty), D(\mathcal{A}))$.*

In addition we have the following characterization of the space $D(\mathcal{A})$ (see [39])

Proposition 2.2.2. *Let $u = (y, z) \in \mathcal{H}$. Then $u \in D(\mathcal{A})$ if and only if the following condition holds*

$$\begin{cases} y \in H^3(0,1) \bigcap W, \\ z \in W, \\ y_{xx}(1) = 0. \end{cases} \tag{2.2.11}$$

In particular, the resolvent $(I - \mathcal{A})^{-1}$ of $\mathcal{A}$ is compact on the energy space $\mathcal{H}$ and the solution of the system (2.1.1)-(2.1.4) satisfies

$$y(t) \in \mathcal{C}^2([0,\infty), V) \cap \mathcal{C}^1([0,\infty), W) \cap \mathcal{C}^0([0,\infty), H^3(0,1) \cap W)). \tag{2.2.12}$$

Our goal is to establish a polynomial energy decay rate via an observability inequality for the conservative problem by a method introduced in [4]. Then we give the following characterization of the linear bounded operator $\mathcal{B}_\beta$.

Proposition 2.2.3. *Let $\varphi_0(x) = \gamma^{-1/2} \cosh^{-1}(\gamma^{-1/2}) \sinh(\gamma^{-1/2} x)$ and define the linear bounded operator B by*

$$B : \mathbb{C} \to V, \quad \text{such} \quad \text{that} \quad B1 = \varphi_0. \tag{2.2.13}$$

Then we have

1. *$C\varphi_0 = \delta_1$, where δ_1 is the Dirac distribution at $x = 1$ and C defined in (2.2.5).*

2. *For all $y \in V$, $B^* y = y(1)$ where B^* is the adjoint operator of B with respect to the pivot space V.*

3. *For all $y \in V$, $C^{-1}\widetilde{B} y = BB^* y$.*

Proof. **(1)** Let $\varphi \in V$, using (2.2.4)-(2.2.5), we get

$$< C\varphi_0, \varphi >_{V'\times V} = (\varphi_0, \varphi)_V = \int_0^1 (\varphi_0\varphi + \gamma\varphi_{0x}\varphi_x)dx = \int_0^1 (\varphi_0 - \gamma\varphi_{0xx})\varphi dx + \varphi(1) = \varphi(1).$$

This leads to the desired equality.

(2) Let $v \in V$, then, using (2.2.13), we have

$$< 1, B^*v >_{\mathbb{C}\times\mathbb{C}} = (B1, v)_V = (\varphi_0, v)_V.$$

On the other hand, we have

$$(\varphi_0, v)_V = \int_0^1 (\varphi_0 v + \gamma\varphi_{0x}v_x)dx = \int_0^1 (\varphi_0 - \gamma\varphi_{0xx})v dx + v(1) = v(1).$$

This implies that $B^*v = v(1)$ for all $v \in V$.

(3) Let $y \in V$. Then using (2.2.4) and (2.2.5), we get

$$(C^{-1}\widetilde{B}y, \varphi)_V = < \widetilde{B}y, \varphi >_{V'\times V} = y(1)\varphi(1) = < y(1)\delta_1, \varphi >_{V'\times V}, \ \ \forall \varphi \in V.$$

This implies that

$$\widetilde{B}y = y(1)\delta_1, \tag{2.2.14}$$

On the other hand, using (1), (2) and (2.2.14), we get

$$CBB^*y = Cy(1)B1 = y(1)Cy_0 = y(1)\delta_1 = \widetilde{B}y. \tag{2.2.15}$$

This leads to the desired equality. □

Using Proposition 2.2.3, we reformulate problem (2.2.6) into the following closed loop system

$$\begin{cases} y_{tt} + Ay + \beta BB^*y_t = 0, \\ y(0) = y_0, \ \ y_t(0) = y_1 \end{cases} \tag{2.2.16}$$

where $A = C^{-1}\widetilde{A}$.

We recall the following stability results (see [39])

Theorem 2.2.4. *Assume that $\beta > 0$ and let γ_0 be the solution of the equation*

$$\sqrt{\gamma_0}\sinh^{-1}(\sqrt{\gamma_0}\pi) = 1. \tag{2.2.17}$$

Then for any $\gamma \geq \gamma_0$ the semigroup of contractions $e^t\mathcal{A}$ is strongly asymptotically stable on the energy space $\mathcal{H}$, i.e. *for any $u_0 \in \mathcal{H}$, we have*

$$\lim_{t\to+\infty} \|e^t\mathcal{A}u_0\|_{\mathcal{H}}^2 = 0. \tag{2.2.18}$$

Remark 2.2.5. Using a numerical program we find an approximate value of γ_0 defined in (2.2.17),

$$\gamma_0 \simeq 0.45001246517627713.$$

2.3 Spectral analysis of the operator $\mathcal{A}_0$

In this section, we give the asymptotic form of eigenvalues and eigenfunctions of the operator $\mathcal{A}_0$. Since $\mathcal{A}_0$ is closed with compact resolvent, then the spectrum of $\mathcal{A}_0$ consists entirely of isolated eigenvalues with finite multiplicities (see [21]). Moreover, it is easy to prove that $\mathcal{A}_0$ is a skew-adjoint operator and $\mu = 0$ is not an eigenvalue of $\mathcal{A}_0$. Also the coefficients of $\mathcal{A}_0$ are real then the eigenvalues appears by conjugate pairs. Then we denote $\sigma(\mathcal{A}_0) = \{\lambda_k = i\mu_k, k \in \mathbb{Z}^*\}$ with $\mu_{-k} = -\mu_k$ and $U_k = (y_k, i\mu_k y_k)$ be an associated eigenvector. First, we have

Proposition 2.3.1. *For any $\gamma \geq \gamma_0$, each $\lambda_k \in \sigma(\mathcal{A}_0)$ is simple.*

Moreover any associated eigenfunction $U_k = (y_k, i\mu_k y_k)$ is such that

$$y_k(1) \neq 0.$$

Proof. Assume that there exists an eigenvalue $\lambda = i\mu$ ($\mu \in \mathbb{R}^*$) of $\mathcal{A}_0$ which is not simple. By the fact that $\mathcal{A}_0$ is skew-adjoint, we deduce that there correspond at

least two independent eigenvectors $U = (y, \lambda y)$ and $\tilde{U} = (\tilde{y}, \lambda \tilde{y})$.
If $y(1) \neq 0$ then $\underline{U} = (\underline{y}, \lambda \underline{y}) = \dfrac{\tilde{y}(1)}{y(1)} U - \tilde{U}$ is also an eigenvector associated to λ and satisfies $\underline{y}(1) = 0$. So we may assume that $y(1) = 0$. From the definition of $\mathcal{B}_\beta$ (see (2.2.9)), we see that $\mathcal{B}_\beta U = (0, 0)$, hence

$$\mathcal{A}U = (\mathcal{A}_0 + \mathcal{B}_\beta)U = \lambda U = i\mu U, \;\; U \neq 0.$$

This implies that $i\mu$ is an eigenvalue of $\mathcal{A}$ and it is a contradiction with Theorem 2.2.4 since $\gamma \geq \gamma_0$.

□

Now, let $\lambda = i\mu$ be an eigenvalue of $\mathcal{A}_0$ and $U = (y, z) \in D(\mathcal{A}_0)$ be an associated eigenfunction. Then we have

$$\begin{cases} z = i\mu y, \\ -C^{-1}\widetilde{A}y = i\mu z. \end{cases} \tag{2.3.1}$$

Then, using (2.2.3) and (2.2.5), we interpret (2.3.1) as the following variational equation:

$$-\int_0^1 y_{xx}\overline{\varphi_{xx}}dx + \mu^2 \int_0^1 (y\overline{\varphi} + \gamma y_x \overline{\varphi_x})dx = 0, \;\; \forall \varphi \in W. \tag{2.3.2}$$

This gives that the function y is determined by the following system:

$$\begin{cases} y_{xxxx} + \gamma\mu^2 y_{xx} - \mu^2 y = 0, \\ y(0) = y_x(0) = y_{xx}(1) = y_{xxx}(1) + \gamma\mu^2 y_x(1) = 0. \end{cases} \tag{2.3.3}$$

We have found that λ is an eigenvalue of $\mathcal{A}_0$ if and only if there is a non trivial solution of (2.3.3). The general solution of (2.3.3) is given by

$$y(x) = \sum_{i=1}^{4} c_i e^{t_i x}, \tag{2.3.4}$$

where $t_1(\mu) = \sqrt{\dfrac{-\gamma\mu^2 - \mu\sqrt{\gamma^2\mu^2+4}}{2}}$, $t_2(\mu) = -t_1(\mu)$, $t_3(\mu) = \sqrt{\dfrac{-\gamma\mu^2 + \mu\sqrt{\gamma^2\mu^2+4}}{2}}$ and $t_4(\mu) = -t_3(\mu)$. Here and below, for simplicity we denote $t_i(\mu)$ by t_i . Thus the boundary conditions in (2.3.3) may be written as the following system:

$$M(\lambda)C(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ t_1 & t_2 & t_3 & t_4 \\ (t_1)^2e^{t_1} & (t_2)^2e^{t_2} & (t_3)^2e^{t_3} & (t_4)^2e^{t_4} \\ h_\lambda(t_1) & h_\lambda(t_2) & h_\lambda(t_3) & h_\lambda(t_4) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0, \qquad (2.3.5)$$

where we have set $h_\lambda(t) = (t^3 + \gamma\mu^2 t)e^t$. Hence a non trivial solution y exists if and only if the determinant of $M(\lambda)$ vanishes. Set $f(\lambda) = detM(\lambda)$, thus the characteristic equation is $f(\lambda) = 0$.

Proposition 2.3.2. *(Asymptotic expansion of λ_k)*
Let $k \in \mathbb{N}^*$ sufficiently large. Then there exists $m \in \mathbb{Z}$ such that the eigenvalue $\lambda_k = i\mu_k$ of $\mathcal{A}_0$ satisfy the following asymptotic expansion

$$\mu_k = \frac{(k+m)\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}} + o(1). \qquad (2.3.6)$$

Proof. The proof is decomposed in two steps. Here and below, for simplicity, we drop the index k .

Step 1. We start by the expansion of t_1 and t_3 :

$$t_1 = i\mu\sqrt{\gamma} + \frac{i}{2\gamma\sqrt{\gamma}\mu} - \frac{5i}{8\gamma^3\sqrt{\gamma}\mu^3} + O(\frac{1}{\mu^5}), \qquad (2.3.7)$$

$$t_3 = \frac{1}{\sqrt{\gamma}} - \frac{1}{2\gamma^2\sqrt{\gamma}\mu^2} + O(\frac{1}{\mu^4}). \qquad (2.3.8)$$

Using (2.3.7)-(2.3.8), we find the asymptotic development of :

$$t_1^2e^{t_1} = e^{i\mu\sqrt{\gamma}}(-\gamma\mu^2 - \frac{i\mu}{2\sqrt{\gamma}} + O(1)), \qquad (2.3.9)$$

$$t_2^2 e^{t_2} = e^{-i\mu\sqrt{\gamma}}(-\gamma\mu^2 + \frac{i\mu}{2\sqrt{\gamma}} + O(1)), \tag{2.3.10}$$

$$t_3^2 e^{t_3} = e^{\frac{1}{\sqrt{\gamma}}}(\frac{1}{\gamma} - \frac{1+2\sqrt{\gamma}}{2\gamma^3\sqrt{\gamma}\mu^2} + O(\frac{1}{\mu^4})), \tag{2.3.11}$$

$$t_4^2 e^{t_4} = e^{-\frac{1}{\sqrt{\gamma}}}(\frac{1}{\gamma} + \frac{1-2\sqrt{\gamma}}{2\gamma^3\sqrt{\gamma}\mu^2} + O(\frac{1}{\mu^4})). \tag{2.3.12}$$

This gives

$$h_\lambda(t_1) = e^{i\mu\sqrt{\gamma}}(-i\frac{\mu}{\sqrt{\gamma}} + \frac{1}{2\gamma^2} + O(\frac{1}{\mu})), \tag{2.3.13}$$

$$h_\lambda(t_2) = e^{-i\mu\sqrt{\gamma}}(i\frac{\mu}{\sqrt{\gamma}} + \frac{1}{2\gamma^2} + O(\frac{1}{\mu})), \tag{2.3.14}$$

$$h_\lambda(t_3) = e^{\frac{1}{\sqrt{\gamma}}}(\sqrt{\gamma}\mu^2 + \frac{\sqrt{\gamma}-1}{2\gamma^2} + O(\frac{1}{\mu^2})), \tag{2.3.15}$$

$$h_\lambda(t_4) = e^{-\frac{1}{\sqrt{\gamma}}}(\sqrt{\gamma}\mu^2 + \frac{\sqrt{\gamma}-1}{2\gamma^2} + O(\frac{1}{\mu^2})). \tag{2.3.16}$$

Combining (2.3.7)-(2.3.16) and (2.3.5), we obtain

$$M(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ i\mu\sqrt{\gamma} + O(\frac{1}{\mu}) & -i\mu\sqrt{\gamma} + O(\frac{1}{\mu}) & \frac{1}{\sqrt{\gamma}} + O(\frac{1}{\mu^2}) & -\frac{1}{\sqrt{\gamma}} + O(\frac{1}{\mu^2}) \\ z_\mu e^{i\mu\sqrt{\gamma}} + O(1) & \overline{z}_\mu e^{-i\mu\sqrt{\gamma}} + O(1) & \frac{1}{\gamma}e^{\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\mu^2}) & \frac{1}{\gamma}e^{-\frac{1}{\sqrt{\gamma}}} + O(\frac{1}{\mu^2}) \\ \frac{-i\mu}{\sqrt{\gamma}}e^{i\mu\sqrt{\gamma}} + O(1) & \frac{i\mu}{\sqrt{\gamma}}e^{-i\mu\sqrt{\gamma}} + O(1) & \sqrt{\gamma}\mu^2 e^{\frac{1}{\sqrt{\gamma}}} + O(1) & -\sqrt{\gamma}\mu^2 e^{-\frac{1}{\sqrt{\gamma}}} + O(1) \end{pmatrix} \tag{2.3.17}$$

where $z_\mu = -\gamma\mu^2 - \frac{i\mu}{2\sqrt{\gamma}}$. Then, after some computations, we find the following asymptotic development of $f(\mu)$ the determinant of $M(i\mu)$:

$$f(\mu) = \mu^5 f_0(\mu) + \mu^4 f_1(\mu) + O(\mu^3)$$

where

$$f_0(\mu) = 4i\gamma^2 \cos(\mu\sqrt{\gamma})\cosh(\frac{1}{\sqrt{\gamma}}), \tag{2.3.18}$$

$$f_1(\mu) = -2i\sqrt{\gamma}\sin(\mu\sqrt{\gamma})(\cosh(\frac{1}{\sqrt{\gamma}}) + 2\sqrt{\gamma}\sinh(\frac{1}{\sqrt{\gamma}})). \tag{2.3.19}$$

Then we set

$$\widetilde{f(\mu)} = \frac{f(\mu)}{\mu^5} = f_0(\mu) + \frac{f_1(\mu)}{\mu} + O(\frac{1}{\mu^2}). \tag{2.3.20}$$

Step 2. We look at the roots of f_0 that we denote by μ_k^0.

Solving $f_0(\mu_k) = 0$, we find

$$\cos(\sqrt{\gamma}\mu_k) = 0.$$

This gives

$$\mu_k^0 = \frac{k\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}}, \ k \in \mathbb{Z}.$$

Now with the help of Rouché's theorem, and for μ large enough, we show that the roots $\widetilde{f}$ are close to those of f_0 and :

$$\mu_k = \frac{k'\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}} + o(1) \quad \text{where} \quad k' = k + m. \tag{2.3.21}$$

□

We will serve the asypmtotic behavior (2.3.21) to provide an estimate on the solution y of initial value problem (2.3.3). Set

$$F_0 = 4i\gamma^2 \cosh\frac{1}{\sqrt{\gamma}} \tag{2.3.22}$$

and

$$F_1 = -2i\sqrt{\gamma}(\cosh\frac{1}{\sqrt{\gamma}} + 2\sqrt{\gamma}\sinh\frac{1}{\sqrt{\gamma}}). \tag{2.3.23}$$

Proposition 2.3.3. *The solution y of the undamped initial value problem (2.3.3) satisfies the estimates*

$$y(1) = -2\frac{\gamma F_1}{F_0}\sinh\frac{1}{\sqrt{\gamma}} - 2\cosh\frac{1}{\sqrt{\gamma}} + o(1), \quad \|y\|_W \sim O(|\mu_k^2|) \ \text{and} \ \|U_k\| \sim O(|\mu_k^2|). \tag{2.3.24}$$

Proof. For clarity, we devide the proof into several steps.

Step 1. There exists a solution $C(\mu_k)$ of $\widetilde{M}(\mu_k)C(\mu_k) = 0$ which has the form :

$$C(\mu_k) = C_0 + O(\frac{1}{|\mu_k|}), \tag{2.3.25}$$

where

$$C_0 = (1, 1, -(\gamma\frac{F_1}{F_0} + 1), (\gamma\frac{F_1}{F_0} - 1)).$$

Let $c_1 = 1$, you see in the proof the validation of this choice, using (2.3.5), we get

$$\begin{cases} c_2 + c_3 + c_4 = -1, \\ c_2 t_2 + c_3 t_3 + c_4 t_4 = -t_1, \\ c_2 t_2^2 e^{t_2} + c_3 t_3^2 e^{t_3} + c_4 t_4^2 e^{t_4} = -t_1^2 e^{t_1}. \end{cases} \tag{2.3.26}$$

Now, using Cramer's rule, we have

$$c_2 = \frac{\alpha_2}{D}, \ \ c_3 = \frac{\alpha_3}{D} \ \text{ and } \ c_4 = \frac{\alpha_4}{D}$$

where

$$\alpha_2 = 2t_1^2 t_3 e^{t_1} - t_3^3(e^{t_3} + e^{-t_3}) - t_1 t_3^2(e^{t_3} + e^{-t_3}), \tag{2.3.27}$$

$$\alpha_3 = t_1^3(e^{t_1} + e^{-t_1}) - t_1^2 t_3(e^{t_1} - e^{-t_1}) - 2t_1 t_3^2 e^{-t_3}, \tag{2.3.28}$$

$$\alpha_4 = -t_1^3(e^{t_1} + e^{-t_1}) - t_1^2 t_3(e^{t_1} - e^{-t_1}) + 2t_1 t_3^2 e^{-t_3}, \tag{2.3.29}$$

and

$$D = -2t_1^2 t_3 e^{-t_1} + t_1 t_3^2(e^{t_3} + e^{-t_3}) - t_3^3(e^{t_3} - e^{-t_3}). \tag{2.3.30}$$

Substitute (2.3.7)-(2.3.12) in (2.3.27), we get

$$\alpha_2 = -2\sqrt{\gamma}\mu_k^2 e^{i\mu_k\sqrt{\gamma}} + O(|\mu_k|). \tag{2.3.31}$$

Then, using (2.3.21) and (3.3.47), we obtain

$$\alpha_2 = \mp i2\sqrt{\gamma}\mu_k^2 + O(|\mu_k|). \tag{2.3.32}$$

Similarly, using (2.3.7)-(2.3.12), (2.3.21) and (2.3.30), we get

$$D = -2i\sqrt{\gamma}\mu_k^2 + O(|\mu_k|). \tag{2.3.33}$$

Then using (2.3.32) and (2.3.33), we get

$$c_2 = \frac{\alpha_2}{D} = 1 + O(\frac{1}{|\mu_k|}). \tag{2.3.34}$$

To find c_3 and c_4, substitute (2.3.21) in (2.3.20) to get

$$\cos\sqrt{\gamma}\mu_k = -\frac{F_1}{F_0}\sin\sqrt{\gamma}\mu_k\frac{1}{\mu_k} + O(\frac{1}{\mu_k^2}),$$

where F_0 and F_1 defined in (2.3.22) and (2.3.23) . Then using (2.3.21), we obtain

$$\cos\sqrt{\gamma}\mu_k = -\epsilon_k\frac{F_1}{F_0}\frac{1}{\mu_k} + O(\frac{1}{\mu_k^2}), \quad \text{where} \quad \epsilon_k = \pm 1. \tag{2.3.35}$$

Using (2.3.7)-(2.3.12) in the determinant α_3, we obtain

$$\alpha_3 = t_1^3(e^{t_1} + e^{-t_1}) - t_1^2 t_3(e^{t_1} - e^{-t_1}) - 2t_1 t_3^2 e^{-t_3}$$

$$= -2i\gamma\sqrt{\gamma}\mu_k^3(\cos\sqrt{\gamma}\mu_k + O(\frac{1}{|\mu_k|})) + 2i\gamma\mu_k^2\frac{1}{\sqrt{\gamma}}(\sin\sqrt{\gamma}\mu_k + O(\frac{1}{|\mu_k|})). \tag{2.3.36}$$

Combining (2.3.35) and the (2.3.36) to obtain the following asymptotic behavior

$$\alpha_3 = 2i\sqrt{\gamma}\mu_k^2\epsilon_k(\gamma\frac{F_1}{F_0} + 1) + o(1). \tag{2.3.37}$$

Finally using (2.3.37) and (2.3.33), we get

$$c_3 = -(\gamma\frac{F_1}{F_0} + 1) + o(1), \tag{2.3.38}$$

Similarly, we find

$$c_4 = \gamma\frac{F_1}{F_0} - 1 + o(1). \tag{2.3.39}$$

Step 2. Estimates of $y(1)$. Using equation (3.3.97), we have

$$y(1) = c_1 e^{t_1} + c_2 e^{t_2} + c_3 e^{t_3} + c_4 e^{t_4}.$$

Then, substitute $C_0 = (1, 1, -(\gamma\frac{F_1}{F_0} + 1), (\gamma\frac{F_1}{F_0} - 1))$ in $y(1)$ and use (2.3.7) and (2.3.8), we get

$$= 2\cos\mu\sqrt{\gamma} - 2\frac{\gamma F_1}{F_0}\sinh\frac{1}{\sqrt{\gamma}} - 2\cosh\frac{1}{\sqrt{\gamma}} + o(1) \tag{2.3.40}$$

$$y(1) = -2\frac{\gamma F_1}{F_0}\sinh\frac{1}{\sqrt{\gamma}} - 2\cosh\frac{1}{\sqrt{\gamma}} + o(1) \neq 0.$$

Step 3. Estimates of $\|y\|_W$. Note that

$$\|y\|_W^2 = \int_0^1 |y_{xx}|^2 dx.$$

We start by

$$\int_0^1 |y_{xx}|^2 dx = \sum_{i=1}^{4}\sum_{j=1}^{4} c_i(\int_0^1 t_i^2 e^{t_i x}\overline{t_j^2 e^{t_j x}} dx)\overline{c_j},$$

$$= C'(\mu_k)G\overline{C'(\mu_k)}$$

where

$$G = (g_{ij})_{i,j=1..4} \quad \text{with} \quad g_{ij} = \int_0^1 e^{(ti+\overline{t_j})x}dx \quad \text{and} \quad C'(\mu_k) = (t_i^2 c_i)_i.$$

A direct computation gives

$$\int_0^1 e^{(t1+\overline{t_1})x}dx = \int_0^1 e^{(t2+\overline{t_2})x}dx = \int_0^1 e^{(t3+\overline{t_4})x}dx = 1. \tag{2.3.41}$$

In addition, for $t_i + \overline{t_j} \neq 0$, we have

$$\int_0^1 e^{(ti+\overline{t_j})x}dx = \frac{e^{t_i+\overline{t_j}}}{t_i+\overline{t_j}} - \frac{1}{t_i+\overline{t_j}}. \tag{2.3.42}$$

Therefore, using (2.3.41) and (2.3.42), we find

$$G = G_0 + O(\frac{1}{\mu_k})$$

where

$$G_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{(e^{\frac{2}{\sqrt{\gamma}}}-1)\sqrt{\gamma}}{2} & 1 \\ 0 & 0 & 1 & \frac{(1-e^{\frac{-2}{\sqrt{\gamma}}})\sqrt{\gamma}}{2} \end{pmatrix}, \tag{2.3.43}$$

and $O(\frac{1}{\mu_k})$ is a matrix with all the entries are $O(\frac{1}{\mu_k})$.
Using (2.3.7)-(2.3.8), we obtain

$$C'(\mu_k) = C_0'(\mu_k) + O(1)$$

where

$$C_0'(\mu_k) = (-\gamma\mu_k^2, -\gamma\mu_k^2, 0, 0).$$

Then we deduce that

$$\int_0^1 |y_{xx}|^2 = (C_0'(\mu_k) + O(1))(G_0 + O(\frac{1}{\mu_k}))(\overline{C_0'(\mu_k) + O(1)})^T$$

$$= C_0'(\mu_k)G_0\overline{C_0'(\mu_k)}^T + O(|\mu_k^3|) \sim O(|\mu_k|^4). \tag{2.3.44}$$

Step 4. Estimates of $\|y\|_V$. Similarly, we prove that

$$\int_0^1 |y|^2 dx \sim O(1), \quad \int_0^1 |y_x|^2 dx \sim O(|\mu_k|^2). \tag{2.3.45}$$

Therefore using (2.3.45), we deduce that

$$\|y\|_V \sim O(|\mu_k|). \tag{2.3.46}$$

Finally using (2.3.44) and (2.3.45), we obtain

$$\|U_k\|_{\mathcal{H}} \sim O(|\mu_k|^2). \tag{2.3.47}$$

□

Eigenvectors of $\mathcal{A}_0$. The set of eigenvectors of $\mathcal{A}_0$ corresponding to μ_k is the set $\{U_k = (y_k, z_k) \in D(\mathcal{A}_0)\}_k$ where U_k has the following form :

$$U_k = \begin{pmatrix} y_k \\ i\mu_k y_k \end{pmatrix}. \tag{2.3.48}$$

For the sequel, it is useful to introduce the set $\{\widetilde{U_k}\}_{k\in\mathbb{Z}^*}$ of normalized eigenvectors of $\mathcal{A}_0$ such that

$$\forall k \in \mathbb{Z}^*,\ \widetilde{U_k} = \frac{1}{\|U_k\|_{\mathcal{H}}} U_k.$$

Remark that if we set $\widetilde{U_k} = (\widetilde{y_k}, \widetilde{z_k})$, then from Proposition 2.3.3 and (2.3.48) we have

$$|\widetilde{y_k}(1)| = O(\frac{1}{|\mu_k|^2}) = O(\frac{1}{|k|^2}),\ \text{and}\ \ |\widetilde{z_k}(1)| = O(\frac{1}{|\mu_k|}) = O(\frac{1}{|k|}). \tag{2.3.49}$$

2.4 Polynomial stability for smooth initial data

We know that the Rayleigh beam equation subject to one boundary control force is strongly but not exponentially stable (see [39]). In this section, our goal is to study the polynomial stability of the energy of the Rayleigh beam equation subject to one boundary control force. Our method uses a methodology introduced by Ammari and Tucsnak in [4], where the polynomial stability for the damped problem is reduced to an observability inequality of the corresponding undamped problem combined to a boundedness property of the transfer function of the associated undamped system. Our main results are the following polynomial-type decay estimation

Theorem 2.4.1. *(Polynomial energy decay rate)*
Assume that $\gamma > \gamma_0$. Then, there exist a constant $c > 0$, such that, for all $t > 0$

and for all $(y^0, y^1) \in D(\mathcal{A})$ *the solution of system (2.2.16) verifying the following estimate*

$$E(y(t)) \leq \frac{c}{(1+t)} \|(y^0, y^1)\|^2_{D(\mathcal{A})}. \tag{2.4.1}$$

First, we will establish an observability inequality for the undamped problem corresponding to (2.2.16)

$$\begin{cases} y_{tt} + Ay = 0, \\ y(0) = y_0, \;\; y_t(0) = y_1. \end{cases} \tag{2.4.2}$$

Lemma 2.4.2. *(Observability estimate)*
Assume that $\gamma > \gamma_0$*. Then, there exist* $T > 0$ *and* $C_T > 0$ *such that the solution of (2.4.2) satisfies*

$$\int_0^T |B^* y_t(t)|^2 dt \geq C_T \|(y^0, y^1)\|^2_{D(\mathcal{A}_0)'} \tag{2.4.3}$$

where $D(\mathcal{A}_0)'$ *is the dual of* $D(\mathcal{A}_0)$ *obtained by means of the inner product in* $\mathcal{H}$*.*

Proof. Let $u = (y, y_t)$, the (2.4.2) is equivalent to following undamped problem

$$\begin{cases} u_t = \mathcal{A}_0 u, \\ u(0) = u_0. \end{cases} \tag{2.4.4}$$

Since $u_0 = (y^0, y^1) \in D(\mathcal{A}_0)$ we can write

$$u(t) = \sum_{k \in \mathbb{Z}^*} u_0^k e^{i\mu_k t} \widetilde{U_k}$$

where $\widetilde{U_k}$ is the normalized eigenvector of the operator $\mathcal{A}_0$. Therefore

$$y_t(t) = z(t) = \sum_{k \in \mathbb{Z}^*} e^{i\mu_k t} u_0^k \widetilde{z}_k.$$

This implies that

$$B^* y_t = z(1,t) = \sum_{k \in \mathbb{Z}^*} e^{i\mu_k t} u_0^k \widetilde{z}_k(1).$$

Using the asymptotic expansion (2.3.6) and the fact that the eigenvalues $i\mu_k$ of $\mathcal{A}_0$ are simple and isolated, we deduce that there exists a constant $\alpha_0 > 0$ such that

$\mu_{k+1} - \mu_k \geq \alpha_0$. Then, from Ingham inequality (see [20]) we deduce that there exist $T > 0$ and $\widetilde{c}(T) > 0$ such that

$$\int_0^T |y_t(1,t)|^2 dt \geq \widetilde{c}(T) \sum_{k \in \mathbb{Z}^*} |u_0^k|^2 |\widetilde{z}_k(1)|^2.$$

On the other hand using (2.3.49), we get

$$\int_0^T |y_t(1,t)|^2 dt \geq c(T) \sum_{k \in \mathbb{Z}^*} |u_0^k|^2 \frac{1}{k^2} = \|u_0\|^2_{D(\mathcal{A}_0)'}.$$

The proof of theorem is completed . □

Next, we will check the boundedness of the following transfer function :

$$H : \mathbb{C}_+ = \{\lambda \in \mathbb{C} | Re\lambda > 0\} \to L(\mathbb{C})$$

$$\lambda \mapsto H(\lambda) = \lambda \beta B^* (\lambda^2 + A)^{-1} B. \tag{2.4.5}$$

Let $\alpha > 0$, we define the set $C_\alpha := \{\lambda \in \mathbb{C} | Re\lambda = \alpha\}$.

Lemma 2.4.3. *Assume that $\gamma > \gamma_0$. The transfer function H defined in (2.4.5) is bounded on C_1.*

Proof. Let $a \in \mathbb{C}$. Using the definition of B, we have

$$(\lambda^2 + A)^{-1} B(a) = a(\lambda^2 + A)^{-1} B1 = a(\lambda^2 + A)^{-1} y_0.$$

On the other hand, we can write

$$y_0 = \sum_{k=1}^{+\infty} \gamma_k \widetilde{y_k} \text{ with} \sum_{k=1}^{+\infty} |\gamma_k|^2 < +\infty. \tag{2.4.6}$$

Indeed, since $y_0 \in V$ then $(0, y_0) \in D(\mathcal{A}_0)$, hence $(0, y_0) = \sum_{k \in \mathbb{Z}^*} y_0^k \widetilde{U_k}$ with $\sum_{k \in \mathbb{Z}^*} |y_0^k \lambda_k|^2 < +\infty$. Therefore we obtain (2.4.6) since $\widetilde{U_k} = (\widetilde{y_k}, i\mu_k \widetilde{y_k}), \forall k \in \mathbb{Z}^*$. Consequently we have

$$(\lambda^2 + A)^{-1} B1 = \sum \frac{\gamma_k}{\mu_k^2 + \lambda^2} \widetilde{y_k}(x).$$

Using the definition of $B^\star$, we get

$$\frac{H(\lambda)}{\lambda} = \beta B^*(\lambda^2 + A)^{-1}B1 = \beta \sum_{k=1}^{+\infty} \frac{\gamma_k}{\mu_k^2 + \lambda^2}\widetilde{y_k}(1). \tag{2.4.7}$$

For now, assume that there exists a constant $c_1 > 0$ such that

$$|\frac{1}{\mu_k^2 + \lambda^2}| \leq \frac{c_1}{|\lambda|}, \ \forall k \in \mathbb{N}^*, \lambda \in C_1. \tag{2.4.8}$$

Substitute (2.4.8) in (2.4.7), we obtain

$$|\frac{H(\lambda)}{\lambda}| \leq \frac{c_1}{|\lambda|} \sum_{k=1}^{+\infty} |\gamma_k||\widetilde{y_k}(1)|.$$

Using Cauchy-Schwartz inequality, we get

$$|\frac{H(\lambda)}{\lambda}| \leq \frac{c_1}{|\lambda|} (\sum_{k=1}^{+\infty} |\gamma_k|^2)^{\frac{1}{2}} (\sum_{k=1}^{+\infty} |\widetilde{y_k}(1)|^2)^{\frac{1}{2}}.$$

Using (2.3.49), (2.4.6), we get

$$|\frac{H(\lambda)}{\lambda}| < +\infty.$$

To complete the proof of the Lemma, we still have (2.4.8) to prove it. Let $\lambda = 1 + iy \in C_1$, then we have

$$|\frac{1}{\mu_k^2 + \lambda^2}| = |\frac{1}{\mu_k^2 + (1+iy)^2}| = |g_1(\mu_k) - 2ig_2(\mu_k)|,$$

where

$$g_1(\mu_k) = \frac{\mu_k^2 + 1 - y^2}{D}, \ \ g_2(\mu_k) = \frac{y}{D} \ \text{ and } \ D = (\mu_k^2 + 1 - y^2)^2 + 4y^2.$$

Now, it's easy to prove that $g_1(\mu_k)$ has a maximum value at $\mu_k = \sqrt{y^2 + 2y - 1}$, then we have

$$|g_1(\mu_k)| \leq \frac{1}{4|y|}. \tag{2.4.9}$$

Similarly, we prove that $g_2(\mu_k)$ has a maximum value at $\mu_k = \sqrt{y^2 - 1}$, then we have

$$|g_2(\mu_k)| \leq \frac{1}{4|y|}. \tag{2.4.10}$$

Then using (2.4.9)-(2.4.10), we obtain

$$|\frac{1}{\mu_k^2+\lambda^2}| = |g_1(\mu_k) - 2ig_2(\mu_k)| \leq |g_1(\mu_k)| + 2|g_2(\mu_k)| \leq \frac{3}{4|y|}.$$

Which satisfies (2.4.8) and the proof is completed . □

Proof of Theorem 2.4.1. The polynomial energy estimate (2.4.1) is obtained by application of Theorem 2.4 in [4] with $Y_1 \times Y_2 = D(\mathcal{A}_0)$, $X_1 \times X_2 = D(\mathcal{A}_0)'$ and $\theta = \frac{1}{2}$.

2.5 Optimal polynomial decay rate

The aim of this section is to prove the following optimality result

Theorem 2.5.1. *The energy decay rate (2.4.1) is optimal in the sense that for any $\varepsilon > 0$, we cannot expect the decay rate $\frac{1}{t^{1+\varepsilon}}$ for all initial data $U_0 \in D(\mathcal{A})$ and for all $t > 0$.*

For the optimality, we search the asymptotic behavior of the eigenvalues of the operator $\mathcal{A}$. Let λ be an eigenvalue of $\mathcal{A}$ and $u = (y, z)$ be an associated eigenfunction, then $\mathcal{A}u = \lambda u$. Equivalently, we have the following system:

$$\begin{cases} y_{xxxx} - \gamma\lambda^2 y_{xx} + \lambda^2 y = 0, \\ y(0) = y_x(0) = y_{xx}(1) = 0, \\ y_{xxx}(1) - \gamma\lambda^2 y_x(1) - \beta\lambda y(1) = 0. \end{cases} \tag{2.5.1}$$

A general solution of (2.5.1) is given by:

$$y(x) = \sum_{i=1}^{4} c_i e^{\tilde{t}_i x},$$

where

$$\widetilde{t_1}(\lambda) = \sqrt{\frac{\gamma\lambda^2 + \lambda\sqrt{\gamma^2\lambda^2 - 4}}{2}}, \widetilde{t_2}(\lambda) = -\widetilde{t_1}(\lambda), \widetilde{t_3}(\lambda) = \sqrt{\frac{\gamma\lambda^2 - \lambda\sqrt{\gamma^2\lambda^2 - 4}}{2}}, \widetilde{t_4}(\lambda) = -\widetilde{t_3}(\lambda).$$

Here and bellow for simplicity, we denote $\widetilde{t_i}(\lambda)$ by $\widetilde{t_i}$.

Thus the boundary conditions may be written as the following equation:

$$N(\lambda)\widetilde{C}(\lambda) = 0, \tag{2.5.2}$$

where

$$N(\lambda) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \widetilde{t_1} & \widetilde{t_2} & \widetilde{t_3} & \widetilde{t_4} \\ \widetilde{t_1}^2 e^{\widetilde{t_1}} & \widetilde{t_2}^2 e^{\widetilde{t_2}} & \widetilde{t_3}^2 e^{\widetilde{t_3}} & \widetilde{t_4}^2 e^{\widetilde{t_4}} \\ k_\lambda \widetilde{t_1} & k_\lambda(\widetilde{t_2}) & k_\lambda(\widetilde{t_3}) & k_\lambda(\widetilde{t_4}) \end{pmatrix}, \quad \widetilde{C}(\lambda) = \begin{pmatrix} \widetilde{c_1} \\ \widetilde{c_2} \\ \widetilde{c_3} \\ \widetilde{c_4} \end{pmatrix}, \tag{2.5.3}$$

where we have set $k_\lambda(t) = (t^3 - \lambda^2\gamma t - \beta\lambda)e^t$. Since $\mathcal{A}$ is closed with compact resolvent, then the spectrum of $\mathcal{A}$ consists entirely of isolated eigenvalues with finite multiplicities (see [21]). Also the coefficients of $\mathcal{A}$ are real then the eigenvalues appears by conjugate pairs. Then we denote $\sigma(\mathcal{A}) = \{\lambda_k,\ k \in \mathbb{Z}^*\}$ and $U_k = (y_k; \lambda_k y_k)$ be an associated eigenvector.

Proposition 2.5.2. *(Asymptotic expansion of spectrum of $\mathcal{A}$)*
Let $k \in \mathbb{N}^*$ sufficiently large. be an eigenvalues of the operator $\mathcal{A}$. Then there exists $m \in \mathbb{Z}$ such that the eigenvalues λ_k of the operator $\mathcal{A}$ satisfy the following asymptotic expansion

$$\lambda_k = i(\frac{(k+m)\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}} - \frac{A}{k} + \frac{8(-1)^k}{\pi^2\gamma^{5/2}\cosh(\gamma^{-1/2})k^2}) - \frac{B}{k^2} + O(\frac{1}{k^3}) \tag{2.5.4}$$

where

$$A = \frac{2 + 4\sqrt{\gamma}\tanh(\gamma^{-1/2})}{\pi\gamma^{3/2}} \quad \text{and} \quad B = \frac{\beta(4\sqrt{\gamma} + 2\tanh(\gamma^{-1/2}))}{\pi^2\gamma^{3/2}}.$$

Proof. . The proof is decomposed in two steps .

Step 1. We start by the expansion of $\widetilde{t_1}$ and $\widetilde{t_3}$

$$\widetilde{t_1} = \lambda\sqrt{\gamma} - \frac{1}{2\lambda\gamma\sqrt{\gamma}} + O(\frac{1}{\lambda^2}), \tag{2.5.5}$$

$$\widetilde{t_3} = \frac{1}{\sqrt{\gamma}} + O(\frac{1}{\lambda^2}). \tag{2.5.6}$$

Using(2.5.5)-(2.5.6), we find the following asymptotic development:

$$\widetilde{t_1}^2 e^{\widetilde{t_1}} = e^{\lambda\sqrt{\gamma}}\Big(\lambda^2\gamma - \frac{\lambda}{2\sqrt{\gamma}} + \frac{2\gamma - 16\gamma^2}{16\gamma^3}\Big) + O(\frac{1}{\lambda}), \tag{2.5.7}$$

$$\widetilde{t_2}^2 e^{\widetilde{t_2}} = e^{-\lambda\sqrt{\gamma}}\Big(\lambda^2\gamma + \frac{\lambda}{2\sqrt{\gamma}} + \frac{2\gamma - 16\gamma^2}{16\gamma^3}\Big) + O(\frac{1}{\lambda}), \tag{2.5.8}$$

$$\widetilde{t_3}^2 e^{\widetilde{t_3}} = \frac{e^{\frac{1}{\sqrt{\gamma}}}}{\gamma} + O(\frac{1}{\lambda}), \tag{2.5.9}$$

$$\widetilde{t_4}^2 e^{\widetilde{t_4}} = \frac{e^{-\frac{1}{\sqrt{\gamma}}}}{\gamma} + O(\frac{1}{\lambda}). \tag{2.5.10}$$

This gives

$$k_\lambda(\widetilde{t_1}) = e^{\lambda\sqrt{\gamma}}\Big(\frac{(-4\sqrt{2}\gamma - 4\sqrt{2}\beta\gamma^{\frac{3}{2}})\lambda}{4\sqrt{2}\gamma^{\frac{3}{2}}} + \frac{4\sqrt{2}\gamma + 4\sqrt{2}\beta\gamma^{\frac{3}{2}}}{8\sqrt{2}\gamma^3}\Big) + O(\frac{1}{\lambda}), \tag{2.5.11}$$

$$k_\lambda(\widetilde{t_2}) = e^{-\lambda\sqrt{\gamma}}\Big(\frac{(4\sqrt{2}\gamma - 4\sqrt{2}\beta\gamma^{\frac{3}{2}})\lambda}{4\sqrt{2}\gamma^{\frac{3}{2}}} + \frac{4\sqrt{2}\gamma - 4\sqrt{2}\beta\gamma^{\frac{3}{2}}}{8\sqrt{2}\gamma^3}\Big) + O(\frac{1}{\lambda}), \tag{2.5.12}$$

$$k_\lambda(\widetilde{t_3}) = e^{\dfrac{1}{\sqrt{\gamma}}}\big(- \lambda^2\sqrt{\gamma} - \beta\lambda + \frac{(-1+\sqrt{\gamma})}{2\gamma^2}\Big) + O(\frac{1}{\lambda}), \tag{2.5.13}$$

$$k_\lambda(\widetilde{t_4}) = e^{-\dfrac{1}{\sqrt{\gamma}}}\big(\lambda^2\sqrt{\gamma} - \beta\lambda + \frac{-(1+\sqrt{\gamma})}{2\gamma^2}\Big) + O(\frac{1}{\lambda}). \tag{2.5.14}$$

Combining (2.5.5)-(2.5.14) and (2.5.2), we obtain an equivalent system for (2.5.2),

$$\widetilde{N}(\lambda)\widetilde{C}(\lambda) = 0,$$

where $\widetilde{N}(\lambda)$ is given by:

$$\widetilde{N}(\lambda)=\begin{pmatrix} 1 & 1 & 1 & 1 \\ \lambda\sqrt{\gamma}-\dfrac{1}{2\lambda\gamma\sqrt{\gamma}}+O(\dfrac{1}{\lambda^2}) & -\lambda\sqrt{\gamma}+\dfrac{1}{2\lambda\gamma\sqrt{\gamma}}+O(\dfrac{1}{\lambda^2}) & \dfrac{1}{\sqrt{\gamma}}+O(\lambda) & -\dfrac{1}{\sqrt{\gamma}}+O(\lambda) \\ e^{\lambda\sqrt{\gamma}}h_1^- +O(\dfrac{1}{\lambda}) & e^{-\lambda\sqrt{\gamma}}h_1^+ +O(\dfrac{1}{\lambda}) & \dfrac{e^{\frac{1}{\sqrt{\gamma}}}}{\gamma}+O(\dfrac{1}{\lambda}) & \dfrac{e^{-\frac{1}{\sqrt{\gamma}}}}{\gamma}+O(\dfrac{1}{\lambda}) \\ e^{\lambda\sqrt{\gamma}}h_2+O(\dfrac{1}{\lambda}) & e^{-\lambda\sqrt{\gamma}}h_3+O(\dfrac{1}{\lambda}) & e^{\dfrac{1}{\sqrt{\gamma}}}h_4+O(\dfrac{1}{\lambda}) & e^{-\dfrac{1}{\sqrt{\gamma}}}h_5+O(\dfrac{1}{\lambda}) \end{pmatrix} \tag{2.5.15}$$

and where $h_1^{\pm}=\Big(\lambda^2\gamma\pm\dfrac{\lambda}{2\sqrt{\gamma}}+\dfrac{\gamma-8\gamma^2}{8\gamma^3}\Big)$, $h_2=\dfrac{1+\beta\gamma^{\frac12}}{\gamma^{\frac12}}(\frac{1}{2\gamma^{\frac32}}-\lambda)$, $h_3=\dfrac{1-\beta\gamma^{\frac12}}{\gamma^{\frac12}}(\frac{1}{2\gamma^{\frac32}}+\lambda)$, $h_4=\big(-\lambda^2\sqrt{\gamma}-\beta\lambda+\dfrac{(-1+\sqrt{\gamma})}{2\gamma^2}\big)$, $h_5=\big(\lambda^2\sqrt{\gamma}-\beta\lambda+\dfrac{-(1+\sqrt{\gamma})}{2\gamma^2}\big)$.

After some computations, we find the following asymptotic development of $g(\lambda)$ determinant of $\widetilde{N}(\lambda)$ divided by λ^5:

$$g(\lambda)=g_0(\lambda)+\frac{g_1(\lambda)}{\lambda}+\frac{g_2(\lambda)}{\lambda^2}+\frac{O(1)}{\lambda^3} \tag{2.5.16}$$

where

$$g_0(\lambda)=4\gamma^2\cosh(\lambda\sqrt{\gamma})\cosh(\frac{1}{\sqrt{\gamma}}), \tag{2.5.17}$$

$$g_1(\lambda)=4\beta\gamma\sqrt{\gamma}\cosh(\lambda\sqrt{\gamma})\sinh(\frac{1}{\sqrt{\gamma}})-2\sqrt{\gamma}\cosh(\frac{1}{\sqrt{\gamma}})\sinh(\lambda\sqrt{\gamma})-4\gamma\sinh(\frac{1}{\sqrt{\gamma}})\sinh(\lambda\sqrt{\gamma}), \tag{2.5.18}$$

and

$$g_2(\lambda)=-8-8\cosh(\frac{1}{\sqrt{\gamma}})\cosh(\lambda\sqrt{\gamma})+\frac{\cosh(\frac{1}{\sqrt{\gamma}})\cosh(\lambda\sqrt{\gamma})}{2\gamma}+\frac{4\cosh(\lambda\sqrt{\gamma})\sinh(\frac{1}{\sqrt{\gamma}})}{\sqrt{\gamma}} \tag{2.5.19}$$
$$-4\beta\sqrt{\gamma}\cosh(\frac{1}{\sqrt{\gamma}})\sinh(\lambda\sqrt{\gamma})-2\beta\sinh(\frac{1}{\sqrt{\gamma}})\sinh(\lambda\sqrt{\gamma}).$$

Step 2. We look at the roots of g_0 that we denote by λ_k^0.

Solving $g_0(\lambda_k)=0$, we find

$$\cosh(\sqrt{\gamma}\lambda_k)=0.$$

It's equivalent to

$$\lambda_k^0 = i(\frac{k\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}}).$$

Now, with the help of Rouché's theorem, and for λ_k large enough, we show that the roots λ_k of g are close to those of g_0. Then we have

$$\lambda_k = i(\frac{k'\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}}) + o(1), \quad \text{where} \quad k' = k + m \tag{2.5.20}$$

Step 3. From Step 2, we can write

$$\lambda_k = i(\frac{k'\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}}) + \epsilon_k, \tag{2.5.21}$$

where $\epsilon_k = o(1)$. Using (2.5.16), we get

$$g_0(\lambda_k) + \frac{g_1(\lambda_k)}{\lambda_k} + \frac{g_2(\lambda_k)}{\lambda_k^2} + \frac{O(1)}{\lambda_k^3} = 0. \tag{2.5.22}$$

Substitute (2.5.20) in (2.5.17), (2.5.18) and (2.5.19) respectively, we get

$$g_0(\lambda_k) = 4i(-1)^k\gamma^{\frac{5}{2}} \cosh(\frac{1}{\sqrt{\gamma}})\epsilon_k + O(\epsilon_k^3), \tag{2.5.23}$$

$$g_1(\lambda_k) = (-1)^k i(-2\sqrt{\gamma}\cosh(\frac{1}{\sqrt{\gamma}}) - 4\gamma\sinh(\frac{1}{\sqrt{\gamma}}) + 4\beta\gamma^2\sinh(\frac{1}{\sqrt{\gamma}})\epsilon_k + O(\epsilon_k^2)) \tag{2.5.24}$$

and

$$g_2(\lambda_k) = -8 - (-1)^k i(4\beta\sqrt{\gamma}\cosh(\frac{1}{\sqrt{\gamma}}) + 2\beta\sinh(\frac{1}{\sqrt{\gamma}})) + O(\epsilon_k). \tag{2.5.25}$$

Therefore,

$$\frac{g_1(\lambda_k)}{\lambda_k} = (-1)^k\frac{\widetilde{A}}{\alpha_k} + O(\frac{\epsilon_k}{k}), \tag{2.5.26}$$

where

$$\widetilde{A} = -2\sqrt{\gamma}\cosh(\frac{1}{\sqrt{\gamma}}) - 4\gamma\sinh(\frac{1}{\sqrt{\gamma}}) \tag{2.5.27}$$

and

$$\alpha_k = \frac{k'\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}}. \tag{2.5.28}$$

And

$$\frac{g_2(\lambda_k)}{\lambda_k^2} = \frac{8 + (-1)^k i\widetilde{B}}{\alpha_k^2} + O(\frac{\epsilon_k}{k^2}) \tag{2.5.29}$$

where

$$\widetilde{B} = 4\beta\sqrt{\gamma}\cosh(\frac{1}{\sqrt{\gamma}}) + 2\beta\sinh(\frac{1}{\sqrt{\gamma}}). \tag{2.5.30}$$

Substitute (2.5.23),(2.5.26) and (2.5.29) in (2.5.22), we obtain

$$(-1)^k i\gamma^{\frac{5}{2}}\cosh(\frac{1}{\sqrt{\gamma}})\epsilon_k + (-1)^k\frac{\widetilde{A}}{\alpha_k} + \frac{8}{\alpha_k^2} + \frac{(-1)^k i\widetilde{B}}{\alpha_k^2} + O(\frac{\epsilon_k}{k}) = 0. \tag{2.5.31}$$

This implies that

$$\epsilon_k = -\frac{(-1)^k\frac{\widetilde{A}}{\alpha_k} + \frac{8}{\alpha_k^2} + \frac{(-1)^k i\widetilde{B}}{\alpha_k^2}}{(-1)^k i\gamma^{\frac{5}{2}}\cosh(\frac{1}{\sqrt{\gamma}})} + O(\frac{\epsilon_k}{k}). \tag{2.5.32}$$

Using equations (2.5.20), (2.5.28) and (2.5.32), we get

$$\Re(\epsilon_k) = -\frac{\widetilde{B}}{\gamma^{\frac{5}{2}}\cosh(\frac{1}{\sqrt{\gamma}})}\frac{\gamma}{k^2} + O(\frac{1}{k^3}). \tag{2.5.33}$$

□

Proof of Theorem 2.5.1. Let $\varepsilon > 0$ and set $l = \dfrac{\varepsilon}{1+\varepsilon}$. For $k \in \mathbb{N}^*$, let λ_k be an eigenvalue of the operator $\mathcal{A}$ and U^k the associated normalized eigenfunction. Consider the following sequences

$$\beta_k = \frac{k\pi}{\sqrt{\gamma}} + \frac{\pi}{2\sqrt{\gamma}} - \frac{A}{k} + \frac{8(-1)^k}{\pi^2\gamma^{5/2}\cosh(\gamma^{-1/2})k^2},$$

$$(U^k) \subset D(\mathcal{A}).$$

Using (2.5.4) we get

$$\lim_{k\to+\infty}\beta_k^{2-2l}\|(i\beta_k - \mathcal{A})U_k\| = 0.$$

By applying of Borichev Theorem (see [16], [13], [43]), we deduce that the trajectory $e^{t\mathcal{A}}u_0$ decays slower than $\dfrac{1}{t^{\frac{1}{2-2l}}}$ on the time $t \to \infty$. Then we cannot expect the energy decay rate $\frac{1}{t^{1+\varepsilon}}$. The proof is thus complete.

Chapter 3

Some stability results of a mindlin-Timoshenko plates in unbounded domain

3.1 Introduction

In this work we consider the internal stabilization of the following Mindlin-Timoshenko set in the domain $\mathbb{R}^2$:

$$J w_{tt} - K \text{div}(\nabla w + u) + b w_t = 0, \tag{3.1.1}$$

$$\rho u_{tt} - D(\frac{1-\mu}{2}\Delta u + \frac{1+\mu}{2}\nabla \text{div} u) + K(\nabla w + u) + a u_t = 0, \tag{3.1.2}$$

in $\mathbb{R}^2 \times (0, +\infty)$, with the initial conditions

$$u(x,0) = u^0(x), u_t(x,0) = u^1(x), w(x,0) = w^0(x), w_t(x,0) = w^1(x), \quad \forall x \in \mathbb{R}^2 \tag{3.1.3}$$

where J and ρ are two constants depend on the mass per unit of surface area and the (uniform) plate thickness, K is the shear modulus, D is the modulus of flexural rigidity, μ is Poisson's ratio ($0 < \mu < 1$ in physical situations) $a > 0$ and $b > 0$ are constants. The scalar variable $w(x,t)$ represents the displacement of the plate in the vertical direction, while the vectorial variable $u = (u_i)_{i=1}^2$ is the angles of rotation of a filament of the plate (for more details see [26], [27]).

Let (u, w) be a regular solution of system (3.1.1)-(3.1.3), then the natural energy associated is given by:

$$E(t) = \int_{\mathbb{R}^2} (D\frac{1-\mu}{2}|\nabla u|^2 + D\frac{1+\mu}{2}|\text{div}u|^2 + \rho|v|^2 + J|y|^2 + K|\nabla w + u)|^2)dx. \quad (3.1.4)$$

In [14], Belkacem and Kasimov studied the stability of an one-dimentional Timoshenko system in $\mathbb{R}$ with one distributed temperature or Cattaneo dissipation damping. They proved that the heat dissipation alone is sufficient to stabilize the system in both cases, so that additional mechanical damping is unnecessary. But there is a difference between the Timeshenko system in $\mathbb{R}^2$ and its analogous system in $\mathbb{R}$. In fact, the coupling between the equations of the rotational angles (3.1.2) and the displacement equation (3.1.3) is given by the gradient of the scalar variable $w(x,t)$ and the vectorial variable $u = (u_i)_{i=1}^2$ but in the one-dimensional case the the coupling is given by partial derivatives. For this reason, the stability results are no longer the same and of intrinsic difference. Then the question of how it is possible to stabilize system (3.1.1)-(3.1.1) and find ŞsufficientŤ dissipations to produce stability are interesting and open.

In this paper, we consider a Mindlin-Timoshenko system in domain $\mathbb{R}^2$ with two internal or temperature dissipations laws. First, if the system is subject to two internal damping then, using a direct approach based on the Fourier transform, we establish a polynomial energy decay rate for usual initial data. Next, in the case of two temperature dissipation laws, we prove that the system is unstable.

3.2 Well-posedness of the system.

In this section we study the existence, uniqueness and regularity of solution of system (3.1.1)-(3.1.3). First, we define the energy space $\mathcal{H}$ by:

$$\mathcal{H} = H^1(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2)^2 \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2).$$

For all $U = (u, v, w, y)^\top, U^* = (u^*, v^*, w^*, y^*)^\top \in \mathcal{H}$, the inner product in $\mathcal{H}$ is defined by

$$< U, U^* >_{\mathcal{H}} = \int_{\mathbb{R}}^{2} (D\frac{1-\mu}{2}\nabla u \nabla \overline{u}^* + D\frac{1+\mu}{2} \operatorname{div} u \operatorname{div} \overline{u}^* + \widetilde{\rho} v.\overline{v}^* + Jy\overline{y}^* \quad (3.2.1)$$

$$+K(\nabla w + u).(\nabla \overline{w}^* + \overline{u}^*) + w\overline{w}^*)dx.$$

It is easy to check that the inner product (3.2.1) is equivalent to the usual inner product in $\mathcal{H}$. Next, we define the following linear unbounded operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ by

$$D(\mathcal{A}) = \Big\{ U = (u, v, w, y) \in \mathcal{H}; v \in H^1(\mathbb{R}^2)^2, y \in H^1(\mathbb{R}^2),$$

$$\left(\frac{1-\mu}{2}\Delta u + \frac{1+\mu}{2}\nabla \mathrm{div} u \right) \in L^2(\mathbb{R}^2)^2, \Delta w \in L^2(\mathbb{R}^2) \Big\}.$$

$$\mathcal{A} \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} v \\ \rho^{-1}\left(D\left(\frac{1-\mu}{2}\Delta u + \frac{1+\mu}{2}\nabla \mathrm{div} u \right) - K(\nabla w + u) - av \right) \\ y \\ J^{-1}(K \mathrm{div}(\nabla w + u) - by) \end{pmatrix}. \tag{3.2.2}$$

Then setting $U = (u, u_t, w, w_t)^\top$ we rewrite system (3.1.1)-(3.1.3) into an evolution equation

$$\begin{cases} U_t = \mathcal{A}U, \\ U(0) = U_0 = (u^0, u^1, w^0, w^1)^\top \in \mathcal{H}. \end{cases} \tag{3.2.3}$$

We will prove that the operator $\mathcal{A}$ is the infinitesimal generator of a C^0 semigroup $e^{t\mathcal{A}}$ on $\mathcal{H}$. Thus, we have the following result about existence and uniqueness of solutions.

Theorem 3.2.1. *Let $U_0 = (u^0, u^1, w^0, w^1)^\top \in \mathcal{H}$. Then there exists a unique $U = (u, u_t, w, w_t)^\top$ solution of system (3.2.3) satisfying*

$$U \in \mathcal{C}^0([0,\infty), \mathcal{H}). \tag{3.2.4}$$

Moreover, if $U_0 \in D(\mathcal{A})$, then

$$U \in \mathcal{C}^1([0,\infty), \mathcal{H}) \cap \mathcal{C}^0([0,\infty), D(\mathcal{A})). \tag{3.2.5}$$

Let us note that the operator $\mathcal{A}$ is not maximal-dissipative on $\mathcal{H}$ and $\frac{1}{2b}I$ is a bounded operator on $\mathcal{H}$. Accordingly, it is natural to introduce the operator $\widetilde{\mathcal{A}}$ by

$$D(\widetilde{\mathcal{A}}) = D(\mathcal{A}) \text{ and } \widetilde{\mathcal{A}} = \mathcal{A} - \frac{1}{2b}I.$$

Using is the infinitesimal generator of C_0-semigroup of contractions over $\mathcal{H}$. For that purpose we need the two following lemmas.

Lemma 3.2.2. *The operator $\widetilde{\mathcal{A}}$ is dissipative on the energy space $\mathcal{H}$.*

Proof. Let $U = (u, v, w, y)^\top \in D(\mathcal{A})$. Then, by the definition of $\mathcal{A}$ we may write

$$\begin{aligned}< \mathcal{A}U, U >_{\mathcal{H}} = \int_{\mathbb{R}^2} \Big[& D\frac{1-\mu}{2}\nabla v \cdot \nabla \overline{u} + D\frac{1+\mu}{2}(\mathrm{div} v)(\mathrm{div}\overline{u}) + D\left(\frac{1-\mu}{2}\Delta u + \frac{1+\mu}{2}\nabla \mathrm{div} u\right)\cdot \overline{v} \\ & -K(\nabla w + u)\cdot \overline{v} - a|v|^2 + K\mathrm{div}(\nabla w + u)\overline{y} - b|y|^2 + K(\nabla y + v)\cdot(\nabla \overline{w} + \overline{u}) + y\overline{w}\Big] dx.\end{aligned} \tag{3.2.6}$$

By a classical density argument, it is easy to prove the following generalized Green formula:

$$\int_{\mathbb{R}^2} \left(\frac{1-\mu}{2}\Delta u + \frac{1+\mu}{2}\nabla \mathrm{div} u\right)\cdot \overline{v} dx = -\frac{1-\mu}{2}\int_{\mathbb{R}^2} \nabla u \cdot \nabla \overline{v} dx - \frac{1+\mu}{2}\int_{\mathbb{R}^2} (\mathrm{div} u)(\mathrm{div}\overline{v}) dx \tag{3.2.7}$$

and

$$\int_{\mathbb{R}^2} \operatorname{div}(\nabla w + u)\overline{y}dx = -\int_{\mathbb{R}^2} (\nabla w + u) \cdot \nabla \overline{y}dx. \tag{3.2.8}$$

Now, inserting (3.2.7) and (3.2.8) into (3.2.6) and using Young inequality, we get

$$\Re < \mathcal{A}U, U >_{\mathcal{H}} = -\int_{\mathbb{R}^2} (a|v|^2 + b|y|^2)dx + \int_{\mathbb{R}^2} y\overline{w}dx \leq -\int_{\mathbb{R}^2} (a|v|^2 + \frac{b}{2}|y|^2)dx + \frac{1}{2b}\int_{\mathbb{R}^2} |w|^2 dx. \tag{3.2.9}$$

Finally, from equation (3.2.9) and the definition of the norm in $\mathcal{H}$, we obtain

$$\Re < \widetilde{\mathcal{A}}U, U >_{\mathcal{H}} = \Re < \mathcal{A}U, U >_{\mathcal{H}} - \frac{1}{2b}\|U\|^2_{\mathcal{H}} \leq -\int_{\mathbb{R}^2} (a|v|^2 + \frac{b}{2}|y|^2)dx \leq 0. \tag{3.2.10}$$

The proof is thus complete.

□

Lemma 3.2.3. *The operator $\widetilde{\mathcal{A}}$ is maximal on $\mathcal{H}$ i.e. for all real number $\lambda > 0$, $\lambda I - \widetilde{\mathcal{A}}$ is surjective.*

Proof. Let $\lambda > 0$ be a real number and let $F = (f, g, h, m, l)^\top \in \mathcal{H}$. We look for an element $U = (u, v, w, y, \theta)^\top \in D(\widetilde{\mathcal{A}})$ solution of $\lambda U - \widetilde{\mathcal{A}}U = F$. Equivalently, we consider the following system

$$\begin{cases} v = (\lambda + \frac{1}{2b})u - f, \quad y = (\lambda + \frac{1}{2b})w - h, \\ \rho\lambda^2 u + \widetilde{\rho}(\frac{1}{b}\lambda + \frac{1}{4b^2})u - \Big(D\Big(\frac{1-\mu}{2}\Delta u + \frac{1+\mu}{2}\nabla \operatorname{div} u\Big) - k(\nabla w + u) - a(\lambda + \frac{1}{2b})u\Big) = g_\lambda \in L^2(\mathbb{R} \\ J\lambda^2 w + J(\frac{1}{b}\lambda + \frac{1}{4b^2})w - K\operatorname{div}(\nabla w + u) + b(\lambda + \frac{1}{2b})w = j_\lambda \in L^2(\mathbb{R}^2) \end{cases} \tag{3.2.11}$$

where

$$g_\lambda = \rho(g + (\lambda + \frac{1}{2b})f) + af, \;\; j_\lambda = J(m + (\lambda + \frac{1}{2b})h) + bh.$$

Multiplying the first equation by a test function $\overline{u}^* \in H^1(\mathbb{R}^2)^2$ and the second equation by a test function $\overline{w}^* \in H^1(\mathbb{R}^2)$, integrating in $\mathbb{R}^2$ and using generalized

Green formula (3.2.7) and (3.2.8), we obtain the following weak formulation of system (3.2.11)

$$a_\lambda((u,w),(\overline{u}^*,\overline{w}^*)=\int_{\mathbb{R}^2}\Big[g_\lambda\cdot\overline{u}^*+j_\lambda\overline{w}^*\Big]dx\ \ \forall(\overline{u}^*,\overline{w}^*)\in H^1(\mathbb{R}^2)^2\times H^1(\mathbb{R}^2)\quad(3.2.12)$$

where

$$a_\lambda((u,w),(\overline{u}^*,\overline{w}^*)=\int_{\mathbb{R}^2}\Big(\lambda+\frac{1}{2b})(a+\widetilde{\rho}(\lambda+\frac{1}{2b}))u\cdot\overline{u}^*+(\lambda+\frac{1}{2b})(b+J(\lambda+\frac{1}{2b}))w\overline{w}^*$$

$$+D\frac{1-\mu}{2}\nabla u\cdot\nabla\overline{u}^*+D\frac{1+\mu}{2}(\mathrm{div}u)(\mathrm{div}\overline{u}^*)+k(\nabla w+u)\cdot(\nabla\overline{w}^*+\overline{u}^*).$$

It is easy to check that the sesquilinear form a_λ is continuous and coercive on the space $(H^1(\mathbb{R}^2)^2\times H^1(\mathbb{R}^2))^2$ and the right-hand side of (3.2.12) is continuous linear form on the space $H^1(\mathbb{R}^2)^2\times H^1(\mathbb{R}^2)$. Then thanks to Lax-Milgram Theorem, the variational equation (3.2.12) admits a unique solution $(u,w)\in H^1(\mathbb{R}^2)^2\times H^1(\mathbb{R}^2)$. This solution is a solution of (3.2.11) by taking test functions in the form $(u^*,0)$ with $u^*\in D(\mathbb{R}^2)^2$ and $(0,w^*)$ with $w^*\in D(\mathbb{R}^2)$. This leads to the conclusion by setting $v=(\lambda+\frac{1}{2b})u-f\in H^1(\mathbb{R}^2)^2$, $y=(\lambda+\frac{1}{2b})w-h\in H^1(\mathbb{R}^2)$ and remarking that

$$K\mathrm{div}(\nabla w+u)=J\lambda^2w+b(\lambda+(\lambda+\frac{1}{2b}))w-j_\lambda\in L^2(\mathbb{R}^2),$$

$$D\Big(\frac{1-\mu}{2}\Delta u+\frac{1+\mu}{2}\nabla\mathrm{div}u\Big)=\rho\lambda^2u+K(\nabla w+u)+a(\lambda+(\lambda+\frac{1}{2b}))u-g_\lambda\in L^2(\mathbb{R}^2)^2.$$

The proof is thus complete. □

Proof of Theorem 3.2.1 From Lemma 3.2.2 and Lemma 3.2.3 we deduce that $\widetilde{\mathcal{A}}$ is m-dissipatif operator. Then, thanks to Lumer-Philips Theorem [[38], Theorem 1.4.3], we conclude that $\widetilde{\mathcal{A}}$ generates a C_0-semigroup of contractions on $\mathcal{H}$. On the other hand, since the operator $\frac{1}{2b}I$ is bounded on $\mathcal{H}$, then using Theorem 1.1 Chapter 3 in [38], we deduce that $\mathcal{A}=\widetilde{\mathcal{A}}+\frac{1}{2b}I$ generates a C_0 semigroup on $\mathcal{H}$. The proof is thus complete.

3.3 Polynomial stability result

In this section, using a direct approach based on the Fourier transform we will establish the following polynomial stability estimate:

Theorem 3.3.1. *Assume that $DJ - K\rho \neq 0$. Let $U_0 \in \mathcal{H} \cap L^1(\mathbb{R}^2)^6$. Then, the solution U of problem (3.1.1)-(3.1.3) satisfies the following estimates:*

$$\|U(t)\|_{\mathcal{H}}^2 \lesssim \|\widehat{U}_0\|_{\infty}^2 \frac{1}{t} + \|U_0\|_{\mathcal{H}}^2 e^{-ct},$$

where $\|\widehat{U}_0\|_\infty = \|\widehat{u}_1^0\|_\infty + \|\widehat{u}_2^0\|_\infty + \|\widehat{v}_1^0\|_\infty + \|\widehat{v}_2^0\|_\infty + \|\widehat{w}^0\|_\infty + \|\widehat{y}^0\|_\infty$.

We start by taking the Fourier transform of system (3.2.3). With this goal in mind, we obtain the following ODE system:

$$\begin{cases} \widehat{U}'(\xi,t) = \widehat{\mathcal{A}}(\xi)\widehat{U}(\xi,t), \\ \widehat{U}(\xi,0) = \widehat{U}_0(\xi) = (\widehat{u}_1^0(\xi), \widehat{u}_2^0(\xi), \widehat{v}_1^0(\xi), \widehat{v}_2^0(\xi), \widehat{w}_0(\xi), \widehat{y}_0(\xi)), \end{cases} \tag{3.3.1}$$

where the time derivative is denoted by a prime, $\hat{U} = (\hat{u}, \hat{v}, \hat{w}, \hat{y})^T = (\hat{u_1}, \hat{u_2}, \hat{v_1}, \hat{v_2}, \hat{w}, \hat{y})^T$, and the matrix $\widehat{\mathcal{A}}$ is given by

$$\widehat{\mathcal{A}}(\xi) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ N_{31} & N_{32} & -a & 0 & -iK\rho^{-1}\xi_1 & 0 \\ N_{41} & N_{42} & 0 & -a & -iK\rho^{-1}\xi_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ iKJ^{-1}\xi_1 & iKJ^{-1}\xi_2 & 0 & 0 & -KJ^{-1}|\xi|^2 & -b \end{pmatrix}, \tag{3.3.2}$$

where

$$N_{31} = -\rho^{-1}[D(\frac{1-\mu}{2}|\xi|^2 + \frac{1+\mu}{2}|\xi_1|^2) + K], \tag{3.3.3}$$

$$N_{32} = -\rho^{-1}D(\frac{1+\mu}{2}\xi_1\xi_2), \tag{3.3.4}$$

$$N_{41} = -\rho^{-1} D(\frac{1+\mu}{2}\xi_1\xi_2), \tag{3.3.5}$$

and

$$N_{42} = -\rho^{-1}[D(\frac{1-\mu}{2}|\xi|^2 + \frac{1+\mu}{2}|\xi_2|^2) + K]. \tag{3.3.6}$$

The energy of the system $E(t)$ is equivalent to the following energy functional:

$$\widetilde{E}(t) = \int_{\mathbb{R}^2} [(1+|\xi|^2)|\widehat{u}|^2 + |\widehat{v}|^2 + (1+|\xi|^2)|\widehat{w}|^2 + |\widehat{y}|^2] d\xi_1 d\xi_2. \tag{3.3.7}$$

Our aim is to estimate the components of $\hat{U}$. Then, taking the change of variables in polar coordinates $\xi_1 = r\cos\theta$ and $\xi_2 = r\sin\theta$, in (3.3.1), from (3.3.1)-(3.3.6), we have the following ODE system:

$$\rho\widehat{u}_1'' + a\rho\widehat{u}_1' + [D(\frac{1-\mu}{2} r^2\sin^2\theta + r^2\cos^2\theta) + K]\widehat{u}_1 + D(\frac{1+\mu}{2} r^2\cos\theta\sin\theta)\widehat{u}_2 + iKr\cos\theta\widehat{w} = 0, \tag{3.3.8}$$

$$\rho\widehat{u}_2'' + a\rho\widehat{u}_2' + [D(\frac{1-\mu}{2} r^2\cos^2\theta + r^2\sin^2\theta) + K]\widehat{u}_2 + D(\frac{1+\mu}{2} r^2\cos\theta\sin\theta)\widehat{u}_1 + iKr\sin\theta\widehat{w} = 0, \tag{3.3.9}$$

$$J\widehat{w}''(t) + bJ\widehat{w}'(t) + Kr^2\widehat{w} - iKr(\cos\theta\widehat{u}_1 + \sin\theta\widehat{u}_2) = 0. \tag{3.3.10}$$

Multiplying equation (3.3.8) by $\cos\theta$ and equation (3.3.9) by $\sin\theta$, and take $\varphi = \cos\theta\widehat{u}_1 + \sin\theta\widehat{u}_2$, we obtain the following system:

$$\begin{cases} \varphi''(t) + a\varphi'(t) + (\frac{K}{\rho} + \frac{Dr^2}{\rho})\varphi + \frac{iKr}{\rho}\widehat{w} = 0, \\ \widehat{w}''(t) + b\widehat{w}'(t) + \frac{Kr^2}{J}\widehat{w} - \frac{iKr}{J}\varphi = 0, \end{cases} \tag{3.3.11}$$

with the following initials conditions

$$\varphi(0) = A_1 = \cos\theta\,\widehat{u}_1^0 + \sin\theta\,\widehat{u}_2^0,\ \varphi'(0) = A_2 = \cos\theta\,\widehat{v}_1^0 + \sin\theta\,\widehat{v}_2^0,\ \widehat{w}(0) = \widehat{w}_0 \text{ and } \widehat{w}'(0) = \widehat{y}_0. \tag{3.3.12}$$

First, we decompose the solution $(\varphi, \widehat{w})$ in two solutions:

$$(\varphi, \widehat{w}) = (\varphi_1, \widehat{w}_1) + (\varphi_2, \widehat{w}_2)$$

Where, $(\varphi_1, \widehat{w}_1)$ is the solution of

$$\begin{cases} \varphi_1''(t) + a\varphi_1'(t) + (\frac{K}{\rho} + \frac{Dr^2}{\rho})\varphi_1 = 0, \\ \widehat{w}_1''(t) + b\widehat{w}_1'(t) + \frac{Kr^2}{J}\widehat{w}_1 = 0, \end{cases} \tag{3.3.13}$$

with initials conditions (3.3.12). And $(\varphi_2, \widehat{w}_2)$ is the solution of

$$\begin{cases} \varphi_2''(t) + a\varphi_2'(t) + (\frac{K}{\rho} + \frac{Dr^2}{\rho})\varphi_2 = -\frac{iKr}{\rho}\widehat{w}_1 - \frac{iKr}{\rho}\widehat{w}_2, \\ \widehat{w}_2''(t) + b\widehat{w}_2'(t) + \frac{Kr^2}{J}\widehat{w}_2 = \frac{iKr}{J}\varphi_1 + \frac{iKr}{J}\varphi_2, \end{cases} \tag{3.3.14}$$

with initials conditions

$$\varphi_2(0) = \varphi_2'(0) = \widehat{w}_2(0) = \widehat{w}_2'(0) = 0. \tag{3.3.15}$$

To obtain our goal in mind, we need the series of the following lemmas. To simplify, the notation $A \lesssim B$ and $A \sim B$ means the existence of positive constants C_1 and C_2, which are independent of A and B such that $A \leq C_2 B$ and $C_1 B \leq A \leq C_2 B$. In the present, we define the following functions

$$\phi(r) := \min(1, r^2) \text{ and } \psi(r) := \max(1, r) \ \forall r > 0. \tag{3.3.16}$$

Lemma 3.3.2. *Let $(\varphi_1, \widehat{w}_1)$ be the solution of (3.3.13), then we have the following estimates*

$$|\varphi_1(t)| \lesssim |\widehat{u}_1^0| + |\widehat{u}_2^0| e^{-ct} + \frac{|\widehat{v}_1^0| + |\widehat{v}_2^0|}{\psi(r)} e^{-ct}, \tag{3.3.17}$$

$$|\widehat{w}_1(t)| \lesssim |\widehat{w}_0| e^{-c\phi(r)t} + \frac{|\widehat{y}_0|}{\psi(r)} e^{-c\phi(r)t}. \tag{3.3.18}$$

Proof. Since the problem (3.3.13) could be decoupled, we solve separately the two equations. We start by solving the first equation. Its characteristic equation is given by:

$$X^2 + aX + (\frac{K}{\rho} + \frac{Dr^2}{\rho}) = 0. \tag{3.3.19}$$

Solving (3.3.19), we find two solutions

$$X_1 = \frac{-a - \sqrt{\delta_1}}{2} \text{ and } X_2 = \frac{-a + \sqrt{\delta_1}}{2}, \tag{3.3.20}$$

where

$$\delta_1 = a^2 - \frac{4}{\rho}(K + Dr^2). \tag{3.3.21}$$

Thus, we have the following solution:

$$\varphi_1(t) = A_1 h_1(t) + A_2 h_2(t) \tag{3.3.22}$$

where

$$h_1(t) = \frac{X_1 e^{tX_2} - X_2 e^{tX_1}}{X_1 - X_2} \text{ and } h_2(t) = \frac{e^{tX_1} - e^{tX_2}}{X_1 - X_2}. \tag{3.3.23}$$

Then, applying Lemma 3.5.2 with $H_1(t) = \dfrac{h_1(t)}{A_1}$ and $H_2(t) = \dfrac{h_2(t)}{A_2}$, we obtain the desired estimate (3.3.17). Similarly, we find the characteristic equation of the second equation of (3.3.13):

$$Z^2 + bZ + \frac{Kr^2}{J} = 0. \tag{3.3.24}$$

The roots of (3.3.24) are given by:

$$Z_1 = \frac{-b - \sqrt{\delta_2}}{2} \text{ and } Z_2 = \frac{-b + \sqrt{\delta_2}}{2} \tag{3.3.25}$$

where

$$\delta_2 = b^2 - 4\frac{Kr^2}{J}. \tag{3.3.26}$$

Therefore, we have

$$\widehat{w}_1(t) = B_1 k_1(t) + B_2 k_2(t) \tag{3.3.27}$$

where

$$k_1(t) = \frac{Z_1 e^{tZ_2} - Z_2 e^{tZ_1}}{Z_1 - Z_2} \text{ and } k_2(t) = \frac{e^{tZ_1} - e^{tZ_2}}{Z_1 - Z_2}. \tag{3.3.28}$$

Finally, applying Lemma **??** with $K_1(t) = \dfrac{k_1(t)}{B_1}$ and $K_2(t) = \dfrac{k_2(t)}{B_2}$, we obtain the desired estimate (3.3.18). The proof is thus completed . □

Similarly, again the solution of problem (3.3.14) is cut into two,

$$(\varphi_2, \widehat{w}_2) = (\varphi_3, \widehat{w}_3) + (\varphi_4, \widehat{w}_4), \tag{3.3.29}$$

where, $(\varphi_3, \widehat{w}_3)$ is the solution of

$$\begin{cases} \varphi_3''(t) + a\varphi_3'(t) + (\frac{K}{\widetilde{\rho}} + \frac{Dr^2}{\widetilde{\rho}})\varphi_3 = -\frac{iKr}{\widetilde{\rho}}\widehat{w}_1 - \frac{iKr}{\widetilde{\rho}}\widehat{w}_3, \\ \widehat{w}_3''(t) + b\widehat{w}_3'(t) + \frac{Kr^2}{J}\widehat{w}_3 = \frac{iKr}{J}\varphi_3 \end{cases} \tag{3.3.30}$$

and, $(\varphi_4, \widehat{w}_4)$ is the solution of

$$\begin{cases} \varphi_4''(t) + a\varphi_4'(t) + (\frac{K}{\widetilde{\rho}} + \frac{Dr^2}{\widetilde{\rho}})\varphi_4 = -\frac{iKr}{\widetilde{\rho}}\widehat{w}_4, \\ \widehat{w}_4''(t) + b\widehat{w}_4'(t) + \frac{Kr^2}{J}\widehat{w}_4 = \frac{iKr}{J}\varphi_1 + \frac{iKr}{J}\varphi_4. \end{cases} \tag{3.3.31}$$

Furthermore, we also estimate $\varphi_3(t)$ and $\widehat{w}_3(t)$. The following holds.

Lemma 3.3.3. *Assume that $DJ - K\rho \neq 0$ and let $(\varphi_3, \widehat{w}_3)$ be the solution of problem (3.3.30). Then we have the following estimates:*

$$|\varphi_3(t)| \lesssim \sqrt{\phi(r)}\Big[|\widehat{w}_0| + \frac{|\widehat{y}_0|}{\psi(r)}\Big]e^{-c\phi^2(r)t}, \tag{3.3.32}$$

$$|\widehat{w}_3(t)| \lesssim \Big[|\widehat{w}_0| + \frac{\widehat{y}_0}{\psi(r)}\Big]e^{-c\phi^2(r)t}. \tag{3.3.33}$$

Proof. Applying Laplace transform to the problem (3.3.30) we obtain the following algebraic system:

$$\begin{cases} p_1(\lambda)\widehat{\varphi}_3 = -\frac{iKr}{\widetilde{\rho}}\widehat{\widehat{w}_1} - \frac{iKr}{\widetilde{\rho}}\widehat{\widehat{w}_3}, \\ p_2(\lambda)\widehat{\widehat{w}_3} = \frac{iKr}{J}\widehat{\varphi}_3 \end{cases} \tag{3.3.34}$$

where

$$p_1(\lambda) = \lambda^2 + a\lambda + (\frac{K}{\rho} + \frac{Dr^2}{\rho}) \text{ and } p_2(\lambda) = \lambda^2 + b\lambda + \frac{Kr^2}{J}. \tag{3.3.35}$$

It follows that

$$\widehat{\varphi}_3(\lambda) = -i\frac{Kr}{\rho}\frac{p_2(\lambda)}{p_3(\lambda)}\widehat{\widehat{w}_1} \tag{3.3.36}$$

where

$$p_3(\lambda) = p_1(\lambda)p_2(\lambda) - \frac{K^2r^2}{\rho J}. \tag{3.3.37}$$

Now, we will find the estimate of $\varphi_3(t)$. Then, we distinguish two cases:

Case 1. For r Large. First, divide $p_3(\lambda)$ by r^4 then make the change of variable $\widetilde{z} = \frac{\lambda}{r}$, we obtain:

$$\widetilde{p_3}(\widetilde{z}) = q_0(\widetilde{z}) + q_1(\widetilde{z}) \tag{3.3.38}$$

where

$$\begin{aligned} q_0(\widetilde{z}) &= \widetilde{z}^4 + (\frac{D}{\rho} + \frac{K}{J})\widetilde{z}^2 + \frac{DK}{\rho J}, \\ q_1(\widetilde{z}) &= \frac{a+b}{r}\widetilde{z}^3 + \frac{1}{r^2}(ab + \frac{K}{\rho})\widetilde{z}^2 + (\frac{bK}{\rho r^3} + \frac{1}{r}(\frac{bD}{\rho} + \frac{aK}{J}))\widetilde{z}. \end{aligned} \tag{3.3.39}$$

Solving equation $q_0(\widetilde{z}) = 0$, we find four roots

$$\pm i\sqrt{\frac{K}{J}} \text{ and } \pm i\sqrt{\frac{D}{\rho}}.$$

Then using Rouche's theorem, we easily check that $\widetilde{p_3}$ admits four roots $\widetilde{z}_1, \overline{\widetilde{z}_1}, \widetilde{z}_2, \overline{\widetilde{z}_2}$, such that

$$\widetilde{z}_1 = i\sqrt{\frac{K}{J}} + o(1),\ \widetilde{z}_2 = i\sqrt{\frac{D}{\rho}} + o(1).$$

In order to get a better asymptotic behavior of $\widetilde{z}_1$ we can also write

$$\widetilde{z}_1 = i\sqrt{\frac{K}{J}} - \frac{b}{2r} - \frac{i}{2}(\frac{K^{\frac{3}{2}}\sqrt{J}}{DJ - K\rho} + \frac{b^2}{4}\sqrt{\frac{J}{K}})\frac{1}{r^2} + e(r),$$

where $e(r) = o(1)$. Inserting the previous expression of $\widetilde{z}_1$ in the equation $\widetilde{p_3}(\widetilde{z}_1) = 0$ and making the expansion of the left hand side of this equation we get after some computations

$$\frac{2i\sqrt{K}(DJ - K\rho)}{J^{3/2}\rho}e(r) + o(e(r)) + \frac{i(a-b)K^{5/2}}{\sqrt{J}(DJ - K\rho)r^3} + o(\frac{1}{r^3}) = 0.$$

That prove that $e(r) = O(\frac{1}{r^3})$. Similarly we can prove that

$$\widetilde{z}_2 = i\sqrt{\frac{D}{\rho}} - \frac{a}{2r} + e(r)$$

where $e(r) = O(\frac{1}{r^2})$. Consequently we deduce that p_3 has four roots $z_1, \overline{z}_1, z_2$ and $\overline{z}_2$ where

$$z_1 = r(i\sqrt{\frac{K}{J}} - \frac{b}{2r} - \frac{i}{2}(\frac{K^{\frac{3}{2}}\sqrt{J}}{DJ - K\rho} + \frac{b^2}{4}\sqrt{\frac{J}{K}})\frac{1}{r^2} + O(\frac{1}{r^3})) \tag{3.3.40}$$

and

$$z_2 = r(i\sqrt{\frac{D}{\rho}} - \frac{a}{2r} + O(\frac{1}{r^2})). \tag{3.3.41}$$

Using singularity of $\frac{p_2(z)}{p_3(z)}$, we can write for r large

$$\frac{p_2(z)}{p_3(z)} = \frac{\alpha_1}{z - z_1} + \frac{\overline{\alpha_1}}{z - \overline{z_1}} + \frac{\alpha_2}{z - z_2} + \frac{\overline{\alpha_2}}{z - \overline{z_2}}, \ \ \alpha_i \in \mathbb{C}, \ \ i = 1, 2. \tag{3.3.42}$$

Now, we will find α_i for $i \in \{1, 2\}$ the residues of $\frac{p_2(z)}{p_3(z)}$ at z_i where

$$\alpha_i = \frac{p_2(z_i)}{p_3'(z_i)}. \tag{3.3.43}$$

Since $DJ - K\rho \neq 0$, then substitute (3.3.40) in (3.3.35) and the derivative of (3.3.37), we find

$$p_2(z_1) = \frac{K^2}{DJ - K\rho} + O(\frac{1}{r}) \ \text{ and } \ p_3'(z_1) = \frac{2i\sqrt{K}(DJ - K\rho)}{J^{\frac{3}{2}}\rho} r^3 + O(r^2). \tag{3.3.44}$$

Thus by (3.3.43) it follows that

$$\alpha_1 = -i\frac{J^{\frac{3}{2}}K^{\frac{3}{2}}\rho}{2(DJ - K\rho)^2}\frac{1}{r^3} + O(\frac{1}{r^4}) \ \text{ and } \ \overline{\alpha_1} = i\frac{J^{\frac{3}{2}}K^{\frac{3}{2}}\rho}{2(DJ - K\rho)^2}\frac{1}{r^3} + O(\frac{1}{r^4}). \tag{3.3.45}$$

Similarly substitute (3.3.41) in (3.3.35) and the derivative of (3.3.37), we find

$$p_2(z_2) = (-\frac{D}{\rho} + \frac{K}{J})r^2 + O(r) \ \text{ and } \ p_3'(z_2) = 2i(\frac{\sqrt{D}(K\rho - DJ)}{\rho^{\frac{3}{2}}J} r^3) + O(r^2). \tag{3.3.46}$$

Therefore

$$\alpha_2 = -\frac{i}{2r}\sqrt{\frac{\rho}{D}} + O(\frac{1}{r^2}) \text{ and } \overline{\alpha_2} = \frac{i}{2r}\sqrt{\frac{\rho}{D}} + O(\frac{1}{r^2}). \tag{3.3.47}$$

On the other hand, using Lemma 3.5.4 we deduce that all singularities of $\frac{p_2(z)}{p_3(z)}$ have negative real part, then from (3.3.36), we get

$$\varphi_3(t) = \frac{-iKr}{\rho}\mathfrak{L}^{-1}(\frac{p_2}{p_3})(t) * \widehat{w_1}(t). \tag{3.3.48}$$

Now we will find the estimate of $\mathfrak{L}^{-1}\frac{p_2}{p_3}(t)$, where

$$\mathfrak{L}^{-1}\frac{p_2}{p_3}(t) = \sum_{i=1}^{4} \alpha_i e^{z_i t}. \tag{3.3.49}$$

Using (3.3.40),(3.3.41),(3.3.45) and (3.3.47) it is easy to see that

$$\begin{aligned} |\alpha_1 e^{z_1 t}| &\lesssim \frac{1}{r^3} e^{-\frac{b}{2}t} + O(\frac{1}{r^2}) \text{ and } |\overline{\alpha_1} e^{\overline{z_1} t}| \lesssim \frac{1}{r^3} e^{-\frac{b}{2}t} + O(\frac{1}{r^2}), \\ |\alpha_2 e^{z_2 t}| &\lesssim \frac{1}{r} e^{-\frac{a}{2}t} + O(\frac{1}{r}) \text{ and } |\overline{\alpha_2} e^{\overline{z_2} t}| \lesssim \frac{1}{r} e^{-\frac{a}{2}t} + O(\frac{1}{r}). \end{aligned} \tag{3.3.50}$$

It follows that

$$|\mathfrak{L}^{-1}\frac{p_2}{p_3}(t)| \lesssim \frac{1}{r} e^{-ct}. \tag{3.3.51}$$

Finally, applying Lemma 3.5.5 with $f(t) = r\mathfrak{L}^{-1}\frac{p_2}{p_3}(t)$ and $g(t) = \dfrac{\widehat{w_1}(t)}{|\widehat{w_0}| + \frac{|\widehat{y_0}|}{\psi(r)}}$.

Then, from (3.3.48) we obtain the following estimate for r large

$$|\varphi_3(t)| \lesssim (|B_1| + \frac{|B_2|}{\psi(r)})e^{-c\phi(r)t}. \tag{3.3.52}$$

Case 2. For r Small. For r small, we can write

$$p_3(\lambda) = p_0(\lambda) + r^2\widetilde{p_0}(\lambda) \tag{3.3.53}$$

where

$$p_0(\lambda) = \lambda(\lambda + b)(\lambda^2 + a\lambda + K) \text{ and } \widetilde{p_0}(\lambda) = (\frac{D}{\rho} + \frac{K}{J})\mu^2 + (\frac{bD}{\rho} + \frac{aK}{J})\mu + \frac{DK}{\rho J}r^2.$$

Our goal in mind is to find an estimate of $\varphi_3(t)$ for r small. We start by finding $\widetilde{\mu_i}$ $i \in \{1,2,3,4\}$ roots of $p_0(\lambda)$ by:

$$\widetilde{\mu_1} = 0,\ \widetilde{\mu_2} = -b,\ \widetilde{\mu_3} = \frac{-a\rho + \sqrt{a^2\rho^2 + 4\rho K}}{2},\ \widetilde{\mu_4} = \frac{-a\rho + \sqrt{a^2\rho^2 - 4\rho K}}{2}.$$

By a similar way used for r large, using Rouche's theorem, we find μ_i $i \in \{1,2,3,4\}$ roots of $p_3(\lambda)$ for r small:

$$\mu_1 = -\frac{D}{bJ}r^4 + O(r^5) \ \text{ and } \ \mu_2 = -b + O(r^2),$$

$$\mu_3 = \frac{-a\rho + \sqrt{a^2\rho^2 - 4\rho K}}{2} + O(r) \ \text{ and } \ \mu_4 = \frac{-a\rho - \sqrt{a^2\rho^2 - 4\rho K}}{2} + O(r). \tag{3.3.54}$$

Now, we follow same steps as before to find

$$\frac{p_2(\mu)}{p_3(\mu)} = \sum_{i=1}^{4} \frac{\beta_i}{\mu - \mu_i} \tag{3.3.55}$$

where the coefficients $\beta_i = \dfrac{p_2(\mu_i)}{p_3'(\mu_i)}$ are given by:

$$\beta_1 = \frac{\rho}{bJ}r^2 + O(r^4),\ \beta_2 = -\frac{\rho K^2}{bJ(-b^2\rho + ab\rho - K)^2}r^2 + O(r^4),\ \beta_3 = C_1 + O(r^2) \ \text{ and } \ \beta_4 = C_2 + O(r^2). \tag{3.3.56}$$

Using (3.3.56) and (3.3.54), we obtain

$$|\beta_1 e^{\mu_1 t}| \lesssim r^2 e^{-\frac{bJ}{4D}r^4 t},\ |\beta_2 e^{\mu_2 t}| \lesssim r^2 e^{-bt},\ |\beta_3 e^{\mu_3 t}| \lesssim e^{-ct} \ \text{ and } \ |\beta_4 e^{\mu_4 t}| \lesssim e^{-ct}. \tag{3.3.57}$$

It follows that there exists $c > 0$ such that

$$|\mathfrak{L}^{-1}\frac{P_2}{P_3}(t)| = |\sum_{i=1}^{4} \beta_i e^{\mu_i t}| \lesssim r^2 e^{-cr^4 t}. \tag{3.3.58}$$

Finally, substitute (3.3.18),(3.3.58) in (3.3.48) and using Lemma 3.5.5, we get for r small

$$|\varphi_3(t)| \lesssim r\Big[|B_1| + |B_2|\Big]e^{-cr^4 t}. \tag{3.3.59}$$

Now to find the estimate of $\widehat{w}_3(t)$, we will use problem (3.3.30) where

$$\widehat{w}_3(t) = i\frac{Kr}{J}(K_2 * \varphi_3)(t). \tag{3.3.60}$$

Finally, using (3.5.10), (3.3.52) and (3.5.5), we obtain for r large

$$|\widehat{w}_3(t)| \lesssim \Big[|B_1| + \frac{|B_2|}{\psi(r)}\Big]e^{-c\phi(r)t}. \tag{3.3.61}$$

And, using (3.5.10), (3.3.59) and (3.5.5), we obtain for r small

$$|\widehat{w}_3(t)| \lesssim \Big[|B_1| + |B_2|\Big]e^{-cr^4 t} \tag{3.3.62}$$

Finally (3.3.52), (3.3.59) and (3.3.12) gives (3.3.32), similarly (3.3.61),(3.3.62) and (3.3.12) gives (3.3.33). The proof is thus complete. □

Lemma 3.3.4. *Assume that $DJ - K\rho \neq 0$ and let $(\varphi_4, \widehat{w}_4)$ be the solution of problem (3.3.31) . Then we have the following estimates:*

$$|\varphi_4(t)| \lesssim \phi(r)\Big[|\widehat{u}_1^0| + |\widehat{u}_2^0| + \frac{|\widehat{v}_1^0| + |\widehat{v}_2^0|}{\psi(r)}\Big]e^{-c\phi^2(r)t}, \tag{3.3.63}$$

$$|\widehat{w}_4(t)| \lesssim \sqrt{\phi(r)}\Big[|\widehat{u}_1^0| + |\widehat{u}_2^0| + \frac{|\widehat{v}_1^0| + |\widehat{v}_2^0|}{\psi(r)}\Big]e^{-c\phi^2(r)t}. \tag{3.3.64}$$

Proof. The Laplace transform of problem (3.3.31) is given by:

$$\begin{cases} \lambda^2\widehat{\varphi}_4 + a\lambda\widehat{\varphi}_4 + (\frac{K}{\rho} + \frac{Dr^2}{\rho})\widehat{\varphi}_4 = -\frac{iKr}{\rho}\widehat{\widehat{w}_4}, \\ \lambda^2\widehat{\widehat{w}_4} + b\lambda\widehat{\widehat{w}_4}(t) + \frac{Kr^2}{J}\widehat{\widehat{w}_4} = \frac{iKr}{J}\widehat{\varphi}_1 + \frac{iKr}{J}\widehat{\varphi}_4, \end{cases} \tag{3.3.65}$$

it is of the form

$$\begin{cases} P_1(\lambda)\widehat{\varphi}_4 = -\frac{iKr}{\rho}\widehat{\widehat{w}_4}, \\ P_2(\lambda)\widehat{\widehat{w}_4} = \frac{iKr}{J}\widehat{\varphi}_1 + \frac{iKr}{J}\widehat{\varphi}_4, \end{cases} \tag{3.3.66}$$

Where $P_1(\lambda)$ and $P_2(\lambda)$ are the polynomials defined in (3.3.65) .

The procedure as before yields

$$\widehat{w}_4 = i\frac{Kr}{J}\mathfrak{L}^{-1}\frac{P_1}{P_3}(t) * \varphi_1(t). \tag{3.3.67}$$

Where $P_3(\lambda)$ is the polynomial defined in (3.3.37) .

Again as before, we distinguish two cases r large and small . Note that we find same singularities z_i and μ_i solutions for $P_3(\lambda)$ for r large(resp. small) .

case 1 : For r large .

We will start by searching the estimate of $\mathfrak{L}^{-1}\frac{P_1}{P_3}(t)$, where

$$\mathfrak{L}^{-1}\frac{P_1}{P_3}(t) = \sum_{i=1}^{4} \alpha_i' e^{z_i t}. \tag{3.3.68}$$

Where

$$\alpha_i' = \frac{P_1(z_i)}{P_3'(z_i)}.$$

As before, we find

$$\begin{aligned} \alpha_1' &= -\frac{i}{2r}\sqrt{\frac{J}{K}} + O(1), \\ \alpha_2' &= -\frac{iJK^2\rho}{2(DJ-K\rho)^2}\sqrt{\frac{\rho}{D}}\frac{1}{r^3} + O(\frac{1}{r^4}). \end{aligned} \tag{3.3.69}$$

Using (3.3.40), (3.3.41) and (3.3.69) in (3.3.68), we get

$$|\mathfrak{L}^{-1}\frac{P_1}{P_3}(t)| \lesssim \frac{1}{r}e^{-ct}. \tag{3.3.70}$$

Then substitute (3.3.70) and (3.3.17) in (3.3.67), we get

$$|\widehat{w}_4(t)| \lesssim \Big[|A_1| + \frac{|A_2|}{\psi(r)}\Big]e^{-ct}. \tag{3.3.71}$$

case 2 : For r small .

Now we will start by the estimate of $\mathfrak{L}^{-1}\frac{P_1}{P_3}(t)$, where

$$\mathfrak{L}^{-1}\frac{P_1}{P_3}(t) = \sum_{i=1}^{4} \beta_i' e^{\mu_i t}. \tag{3.3.72}$$

Where

$$\beta_i' = \frac{P_1(\mu_i)}{P_3'(\mu_i)}.$$

As before, we find

$$\begin{aligned}
\beta_1' &= \frac{1}{b} + O(r^2),\\
\beta_2' &= -\frac{1}{b} + O(r^2),\\
\beta_3' &= C_1' + O(r^2), where\ C_1'\ is\ a\ constant\ does\ not\ depend\ on\ r\\
\beta_4' &= C_2' + O(r^2), where\ C_2'\ is\ a\ constant\ does\ not\ depend\ on\ r
\end{aligned} \tag{3.3.73}$$

Using (3.3.54) and (3.3.73) in (3.3.72), we get

$$|\mathfrak{L}^{-1}\frac{P_1}{P_3}(t)| \lesssim e^{-cr^4t}. \tag{3.3.74}$$

Then substitute (3.3.74) and (3.3.17) in (3.3.67), we get

$$|\widehat{w}_4(t)| \lesssim r\Big[|A_1| + \frac{|A_2|}{\psi(r)}\Big]e^{-cr^4t}. \tag{3.3.75}$$

And finally, from problem (3.3.31), we have

$$\varphi_4(t) = -\frac{iKr}{\rho}H_2 * \widehat{w}_4(t). \tag{3.3.76}$$

Then using (3.3.71), (3.5.10) and (3.5.5), we find for r large

$$|\varphi_4| \lesssim \Big[|A_1| + \frac{|A_2|}{\psi(r)}\Big]e^{-c\phi(r)t}, \tag{3.3.77}$$

and using (3.3.75), (3.5.10) and (3.5.5), we find for r small

$$|\varphi_4| \lesssim r^2\Big[|A_1| + |A_2|\Big]e^{-cr^4t}. \tag{3.3.78}$$

Finally (3.3.77), (3.3.78) and (3.3.12) gives (3.3.63), similarly (3.3.71), (3.3.75) and (3.3.12) gives (3.3.64). The proof is thus complete. □

Lemma 3.3.5. *$\widehat{u}_1(t)$,$\widehat{u}_2(t)$,$\widehat{v}_1(t)$,$\widehat{v}_2(t)$ and $\widehat{y}(t)$ solutions of (3.3.1) have the following estimations*

$$\begin{cases} |\widehat{u}_1(t)| & \lesssim \Big[\phi(r)[|\widehat{u}_1^0|+|\widehat{u}_2^0|+\dfrac{|\widehat{v}_1^0|+|\widehat{v}_2^0|}{\psi(r)}]+\sqrt{\phi(r)}[|B_1|+\dfrac{|B_2|}{\psi(r)}]\Big]e^{-c\phi^2(r)t}, \\ |\widehat{u}_2(t)| & \lesssim \Big[\phi(r)[|\widehat{u}_1^0|+|\widehat{u}_2^0|+\dfrac{|\widehat{v}_1^0|+|\widehat{v}_2^0|}{\psi(r)}]+\sqrt{\phi(r)}[|B_1|+\dfrac{|B_2|}{\psi(r)}]\Big]e^{-c\phi^2(r)t}, \\ |\widehat{v}_1(t)| & \lesssim \psi(r)\Big[\phi(r)[|\widehat{u}_1^0|+|\widehat{u}_2^0|+\dfrac{|\widehat{v}_1^0|+|\widehat{v}_2^0|}{\psi(r)}]+\sqrt{\phi(r)}[|\widehat{w}_0|+\dfrac{|\widehat{y}_0|}{\psi(r)}]\Big]e^{-c\phi^2(r)t}, \\ |\widehat{v}_2(t)| & \lesssim \psi(r)\Big[\phi(r)[|\widehat{u}_1^0|+|\widehat{u}_2^0|+\dfrac{|\widehat{v}_1^0|+|\widehat{v}_2^0|}{\psi(r)}]+\sqrt{\phi(r)}[|\widehat{w}_0|+\dfrac{|\widehat{y}_0|}{\psi(r)}]\Big]e^{-c\phi^2(r)t}, \\ |\widehat{y}(t)| & \lesssim r\Big[\phi(r)[|\widehat{u}_1^0|+|\widehat{u}_2^0|+\dfrac{|\widehat{v}_1^0|+|\widehat{v}_2^0|}{\psi(r)}]+\sqrt{\phi(r)}[|\widehat{w}_0|+\dfrac{|\widehat{y}_0|}{\psi(r)}]\Big]e^{-c\phi^2(r)t}. \end{cases} \tag{3.3.79}$$

Proof. First we will start by the estimate of $\widehat{u}_1$. For this reason we will use the third row of (3.3.1) by replacing $\sin\theta\widehat{u}_2$ by $\varphi-\cos\theta\widehat{u}_1$, we get the following differential equation with variable $\widehat{u}_1$:

$$\begin{aligned} &\widehat{u}_1''(t)+a\widehat{u}_1'(t)+\rho^{-1}[D\frac{1-\mu}{2}r^2+K]\widehat{u}_1(t)=f_1(t), \\ &\widehat{u}_1(0)=\widehat{u}_1^0,\ \widehat{u}_1'(0)=\widehat{v}_1^0, \end{aligned} \tag{3.3.80}$$

where

$$f_1(t)=-\rho^{-1}D\frac{1+\mu}{2}\cos\theta\varphi(t)-iK\rho^{-1}r\cos\theta w(t). \tag{3.3.81}$$

The solution of (3.3.80) is:

$$\widehat{u}_1(t)=\widehat{u}_1^0H_1(t)+\widehat{v}_1^0H_2(t)+\int_0^t H_2(t-s)f_1(s)ds. \tag{3.3.82}$$

Where $H_i(t)$ are defined in (3.3.23) with different constant .
Using Lemma 3.5.2 we have

$$|\widehat{u}_1^0H_1(t)+\widehat{v}_1^0H_2(t)|\lesssim(|\widehat{u}_1^0|+\frac{|\widehat{v}_1^0|}{\psi(r)})e^{-ct}. \tag{3.3.83}$$

From Theorem 4.1 we have

$$|f_1(t)|\lesssim\psi(r)\Big[\phi(r)[|\widehat{u}_1^0|+|\widehat{u}_2^0|+\frac{|\widehat{v}_1^0|+|\widehat{v}_2^0|}{\psi(r)}]+\sqrt{\phi(r)}[|\widehat{w}_0|+\frac{|\widehat{y}_0|}{\psi(r)}]\Big]e^{-c\phi^2(r)t}. \tag{3.3.84}$$

Therefore using Lemma 3.5.2 and Lemma 3.5.5 we have

$$|\int_0^t H_2(t-s)f(s)ds| \lesssim \Big[\phi(r)[|\widehat{u}_1^0|+|\widehat{u}_2^0|+\frac{|\widehat{v}_1^0|+|\widehat{v}_2^0|}{\psi(r)}]+\sqrt{\phi(r)}[|\widehat{w}_0|+\frac{|\widehat{y}_0|}{\psi(r)}]\Big]e^{-c\phi^2(r)t}. \tag{3.3.85}$$

Combining (3.3.83) and (3.3.85) we find

$$|\widehat{u}_1(t)| \lesssim \Big[\phi(r)[|\widehat{u}_1^0|+|\widehat{u}_2^0|+\frac{|\widehat{v}_1^0|+|\widehat{v}_2^0|}{\psi(r)}]+\sqrt{\phi(r)}[|\widehat{w}_0|+\frac{|\widehat{y}_0|}{\psi(r)}]\Big]e^{-c\phi^2(r)t}. \tag{3.3.86}$$

Now to find the estimates of u_2, we use the fourth row of (3.3.1) by replacing $\cos\theta u_1$ by $\varphi - \sin\theta u_2$ to find a differential equation with variable u_2 :

$$\widehat{u}_2''(t) + a\widehat{u}_2'(t) + \rho^{-1}[D\frac{1-\mu}{2}r^2 + K]\widehat{u}_2(t) = f_2(t), \tag{3.3.87}$$

where

$$f_2(t) = -\rho^{-1}D\frac{1+\mu}{2}\sin\theta\varphi(t) - iK\rho^{-1}r\sin\theta\widehat{w}(t). \tag{3.3.88}$$

And similarly we find an estimation of $u_2(t)$

$$|\widehat{u}_2(t)| \lesssim \Big[\phi(r)[|\widehat{u}_1^0|+|\widehat{u}_2^0|+\frac{|\widehat{v}_1^0|+|\widehat{v}_2^0|}{\psi(r)}]+\sqrt{\phi(r)}[|\widehat{w}_0|+\frac{|\widehat{y}_0|}{\psi(r)}]\Big]e^{-c\phi^2(r)t}. \tag{3.3.89}$$

Secondly, to find the estimate of $v_1(t)$ we derive (3.3.82) and we get

$$\widehat{v}_1(t) = \widehat{u}_1^0 H_1'(t) + \widehat{v}_1^0 H_2'(t) + \int_0^t H_2'(t-s)f_1(s)ds. \tag{3.3.90}$$

As Lemma 3.5.2, we can prove that

$$|H_1'(t)| \lesssim \psi(r)e^{-ct} \text{ and } |H_2'(t)| \lesssim e^{-ct}. \tag{3.3.91}$$

Now substitute (3.3.91) and (3.3.84) in (3.3.90), we obtain

$$|\widehat{v}_1(t)| \lesssim \psi(r)\Big[\phi(r)[|\widehat{u}_1^0|+|\widehat{u}_2^0|+\frac{|\widehat{v}_1^0|+|\widehat{v}_2^0|}{\psi(r)}]+\sqrt{\phi(r)}[|\widehat{w}_0|+\frac{|\widehat{y}_0|}{\psi(r)}]\Big]e^{-c\phi^2(r)t}. \tag{3.3.92}$$

Similarly, we find the estimate of $\widehat{v}_2(t)$

$$|\widehat{v}_2(t)| \lesssim \psi(r)\Big[\phi(r)[|\widehat{u}_1^0|+|\widehat{u}_2^0|+\frac{|\widehat{v}_1^0|+|\widehat{v}_2^0|}{\psi(r)}]+\sqrt{\phi(r)}[|\widehat{w}_0|+\frac{|\widehat{y}_0|}{\psi(r)}]\Big]e^{-c\phi^2(r)t}. \tag{3.3.93}$$

Finally to find the estimate of $y(t)$ we use second equation of (3.3.11) and we obtain the following equation

$$\widehat{w}''(t) + b\widehat{w}'(t) + \frac{Kr^2}{J}\widehat{w}(t) = f_3(t). \tag{3.3.94}$$

Where

$$f_3(t) = iKJ^{-1}r\varphi. \tag{3.3.95}$$

The solution of (3.3.94) is :

$$\widehat{w}(t) = \widehat{w}_0 K_1(t) + \widehat{y}_0 K_2(t) + \int_0^t K_2(t-s) f_3(s) ds. \tag{3.3.96}$$

Now, derive (3.3.96), we obtain

$$\widehat{y}(t) = \widehat{w}_0 K_1'(t) + \widehat{y}_0 K_2'(t) + \int_0^t K_2'(t-s) f_3(s) ds. \tag{3.3.97}$$

As Lemma 3.5.3, we find

$$|K_1'(t)| \lesssim \psi(r)\phi(r)e^{-c\phi(r)t} \text{ and } |K_2'(t)| \lesssim \phi(r)e^{-c\phi(r)t}. \tag{3.3.98}$$

Finally, using theorem 4.1 and (3.3.98) in (3.3.97), we get

$$|y(t)| \lesssim r\Big[\phi(r)[|\widehat{u}_1^0| + |\widehat{u}_2^0| + \frac{|\widehat{v}_1^0| + |\widehat{v}_2^0|}{\psi(r)}] + \sqrt{\phi(r)}[|\widehat{w}_0| + \frac{|\widehat{y}_0|}{\psi(r)}]\Big] e^{-c\phi^2(r)t}. \tag{3.3.99}$$

□

Proof of the main Theorem. We start by defining an energy of $U(t)$ (denoted by $E(t)$) equivalent to $\|U(t)\|^2_{\mathcal{H}}$:

$$E(t) = \|u_1\|^2_{H^1(\mathbb{R}^2)} + \|u_2\|^2_{H^1(\mathbb{R}^2)} + \|v_1\|^2_{L^2(\mathbb{R}^2)} + \|v_2\|^2_{L^2(\mathbb{R}^2)} + \|w\|^2_{H^1(\mathbb{R}^2)} + \|y\|^2_{L^2(\mathbb{R}^2)}, \tag{3.3.100}$$

Now we give an estimate of $E(t)$. First, we have

$$\begin{aligned}
\|u_1\|^2_{H^1(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} (|u_1|^2 + |\partial_x u_1|^2 + |\partial_y u_1|^2) dxdy \\
&= \int_{\mathbb{R}^2} |\widehat{u}_1|^2 + \xi_1^2 |\widehat{u}_1|^2 + \xi_2^2 |\widehat{u}_1|^2 d\xi_1 d\xi_2 \\
&= \int_{\mathbb{R}^2} (1 + |\xi|^2) |\widehat{u}_1(\xi)|^2 d\xi \\
&= \int_{[0,\infty)\times[0,2\pi]} (1 + r^2) |\widehat{u}_1(r,\theta)|^2 r dr d\theta.
\end{aligned} \tag{3.3.101}$$

Substitute (3.3.86) in (3.3.101), we get

$$\begin{aligned}
\|u_1\|^2_{H^1(\mathbb{R}^2)} \lesssim& \int_0^{2\pi} \int_0^{+\infty} (1 + r^2) \Big[\phi^2(r)[|\widehat{u}_1^0|^2 + |\widehat{u}_2^0|^2 + \frac{|\widehat{v}_1^0|^2 + |\widehat{v}_2^0|^2}{\psi^2(r)}] \\
&+\phi(r)[|\widehat{w}_0|^2 + \frac{|\widehat{y}_0|^2}{\psi^2(r)}]\Big] e^{-c\phi^2(r)t} r dr d\theta.
\end{aligned}$$

Then

$$\begin{aligned}
\|u_1\|^2_{H^1(\mathbb{R}^2)} \lesssim& \int_0^{2\pi} \int_0^1 (1 + r^2) \Big[r^4[|\widehat{u}_1^0|^2 + |\widehat{u}_2^0|^2 + |\widehat{v}_1^0|^2 + |\widehat{v}_2^0|^2] + r^2[|\widehat{w}_0|^2 + |\widehat{y}_0|^2]\Big] e^{-cr^4 t} r dr d\theta \\
&+ \int_0^{2\pi} \int_1^{\infty} \Big[(1 + r^2)|\widehat{u}_1^0|^2 + (1 + r^2)|\widehat{u}_2^0|^2 + |\widehat{v}_1^0|^2 + |\widehat{v}_2^0|^2] \\
&+ (1 + r^2)|\widehat{w}_0|^2 + |\widehat{y}_0|^2\Big] e^{-ct} r dr d\theta \\
=& I_1 + I_2.
\end{aligned} \tag{3.3.102}$$

That is, the integral is split into its low-frequency part I_1, and its high-frequency part I_2 .Starting by I_1 we get

$$\begin{aligned}
I_1 \lesssim& \int_0^{2\pi} \int_0^1 (1+r^2)(|\widehat{u}_1^0|^2 + |\widehat{u}_2^0|^2) r^4 e^{-cr^4 t} r dr d\theta + \int_0^{2\pi} \int_0^1 (|\widehat{v}_1^0|^2 + |\widehat{v}_2^0|^2) r^4 e^{-cr^4 t} r dr d\theta. \\
&+ \int_0^{2\pi} \int_0^1 (1 + r^2)|\widehat{w}_0|^2 r^4 e^{-cr^4 t} r dr d\theta + \int_0^{2\pi} \int_0^1 |\widehat{y}_0|^2 r^4 e^{-cr^4 t} r dr d\theta.
\end{aligned}$$

Now suppose that $U_0 \in (L^1(\mathbb{R}^2))^6$ then $\widehat{U}_0 \in L^{+\infty}(\mathbb{R}^2)$ and using the following inequalities

$$\int_0^1 r^5 e^{-cr^4 t} dr \lesssim \frac{1}{t}. \tag{3.3.103}$$

$$\int_0^1 r^3 e^{-cr^4 t} dr \lesssim \frac{1}{t}. \tag{3.3.104}$$

We estimate

$$I_1 \lesssim \|U_0\|_\infty^2 \frac{1}{t}. \tag{3.3.105}$$

For I_2, a direct step give

$$I_2 \lesssim \|U_0\|_{\mathcal{H}}^2 e^{-ct}. \tag{3.3.106}$$

Therefore substitute (3.3.105) and (3.3.106) in (3.3.102), we estimate

$$\|u_1\|_{H^1(\mathbb{R}^2)}^2 \lesssim \|U_0\|_\infty^2 \frac{1}{t} + \|U_0\|_{\mathcal{H}}^2 e^{-ct}. \tag{3.3.107}$$

Similarly, we find

$$\begin{cases} \|u_2\|_{H^1(\mathbb{R}^2)}^2 \lesssim \|U_0\|_\infty^2 \frac{1}{t} + \|U_0\|_{\mathcal{H}}^2 e^{-ct} \\ \|v_1\|_{L^2(\mathbb{R}^2)}^2 \lesssim \|U_0\|_\infty^2 \frac{1}{t} + \|U_0\|_{\mathcal{H}}^2 e^{-ct} \\ \|v_2\|_{L^2(\mathbb{R}^2)}^2 \lesssim \|U_0\|_\infty^2 \frac{1}{t} + \|U_0\|_{\mathcal{H}}^2 e^{-ct} \\ \|w\|_{H^1(\mathbb{R}^2)}^2 \lesssim \|U_0\|_\infty^2 \frac{1}{t} + \|U_0\|_{\mathcal{H}}^2 e^{-ct} \\ \|y\|_{L^2(\mathbb{R}^2)}^2 \lesssim \|U_0\|_\infty^2 \frac{1}{t} + \|U_0\|_{\mathcal{H}}^2 e^{-ct}. \end{cases} \tag{3.3.108}$$

Finally, substitute (3.3.107) and (3.3.108) in (3.3.100), the proof is completed.

3.4 Lack of stability of the Mindlin-Timoshenko plates with heat conduction in unbounded domain

In this section we consider the internal stabilization of the following Mindlin-Timoshenko-Heat model set in the domain $\mathbb{R}^2$:

$$Jw_{tt} - K\text{div}(\nabla w + u) - \alpha\theta_x = 0, \tag{3.4.1}$$

$$\widetilde{\rho}u_{tt} - D(\frac{1-\mu}{2}\Delta u + \frac{1+\mu}{2}\nabla\text{div}u) + K(\nabla w + u) + \delta\nabla\theta = 0, \tag{3.4.2}$$

$$\theta_t - \Delta\theta + \delta\text{div}\, u_t + \alpha w_t = 0, \tag{3.4.3}$$

with the initial conditions

$$u(x,0) = u^0(x), u_t(x,0) = u^1(x), w(x,0) = w^0(x), w_t(x,0) = w^1(x), \theta(x,0) = \theta^0(x) \ \forall x \in \mathbb{R}^2. \tag{3.4.4}$$

where J and ρ are two constants depend on the mass per unit of surface area and the (uniform) plate thickness, K is the shear modulus, D is the modulus of flexural rigidity, μ is Poisson's ratio ($0 < \mu < 1$ in physical situations) α and δ are constants. The scalar variable $w(x,t)$ represents the displacement of the plate in the vertical direction, while the vectorial variable $u = (u_i)_{i=1}^2$ is the angles of rotation of a filament of the plate (for more details see [26], [27]).

Let (u, w) be a regular solution of system (3.4.1)-(3.4.4), then the natural energy associated is given by:

$$E(t) = \int_{\mathbb{R}^2} (D\frac{1-\mu}{2}|\nabla u|^2 + D\frac{1+\mu}{2}|\text{div}u|^2 + \widetilde{\rho}|v|^2 + J|y|^2 + K|\nabla w + u)|^2 + |\theta|^2)dx. \tag{3.4.5}$$

By means of the classical energy method, it is easy to check that

$$E'(t) = -\int_{\mathbb{R}^2} |\nabla\theta|^2 dx. \tag{3.4.6}$$

This formula indicates clearly that the system is dissipative in the sense that its energy is decreasing with respect to the time t. Naturally, one hopes to know if the heat dissipation is strong enough to produce decay of the energy of solutions of system (3.4.1)-(3.4.4) to zero.

We introduce the following Hilbert space

$$\mathcal{H} = H^1(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2)^2 \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$$

with the inner product

$$\begin{aligned} < U, U^* >_{\mathcal{H}} = \int_{\mathbb{R}^2} (D\frac{1-\mu}{2}\nabla u \cdot \nabla \overline{u}^* + D\frac{1+\mu}{2}(\text{div}u)(\text{div}\overline{u}^*) + \\ \rho v.\overline{v}^* + Jy\overline{y}^* + K(\nabla w + u).(\nabla\overline{w}^* + \overline{u}^*) + w\overline{w}^* + \theta\overline{\theta}^*)dx \end{aligned} \tag{3.4.7}$$

for all $U = (u, v, w, y, \theta)^\top, U^* = (u^*, v^*, w^*, y^*, \theta^*)^\top \in \mathcal{H}$. It is easy to check that the inner product (3.4.7) is equivalent to the usual inner product in $\mathcal{H}$. Next, we define the following linear unbounded operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ by

$$D(\mathcal{A}) = \Big\{U = (u, v, w, y, \theta) \in \mathcal{H}; v \in H^1(\mathbb{R}^2)^2, y \in H^1(\mathbb{R}^2), \left(\frac{1-\mu}{2}\Delta u + \frac{1+\mu}{2}\nabla\text{div}u\right) \in L^2(\mathbb{R}^2)^2,$$

$$\mathcal{A}\begin{pmatrix} u \\ v \\ w \\ y \\ \theta \end{pmatrix} = \begin{pmatrix} v \\ \rho^{-1}\left(D\left(\frac{1-\mu}{2}\Delta u + \frac{1+\mu}{2}\nabla\text{div}u\right) - K(\nabla w + u) - \delta\,\nabla\theta\right) \\ y \\ J^{-1}(K\text{div}(\nabla w + u) + \alpha\theta) \\ \Delta\theta - \delta\,\text{div}\,v - \alpha y \end{pmatrix}. \tag{3.4.8}$$

Then setting $U = (u, u_t, w, w_t, \theta)^\top$ we rewrite system (3.4.1)-(3.4.4) into an evolution equation

$$\begin{cases} U_t = \mathcal{A}U, \\ U(0) = U_0 = (u^0, u^1, w^0, w^1, \theta^0)^\top \in \mathcal{H}. \end{cases} \tag{3.4.9}$$

Using the standard semigroup theory, it is easy to show that system (3.4.1)-(3.4.4) is wellposed in the Hilbert space $\mathcal{H}$ (see Theorem 3.2.1).

Our main result is the following

Theorem 3.4.1. *There exists $U_0 \in \mathcal{H}$ such that the energy the solution $U(t)$ of (3.4.9) remains constant, i.e*

$$E(t) = \|U(x,t)\|^2 = E(0), \forall t \geq 0. \tag{3.4.10}$$

We start by taking the Fourier transform of system (3.4.9). With this goal in mind, we obtain the following ODE system:

$$\begin{cases} \widehat{U}'(\xi,t) = \widehat{\mathcal{A}}(\xi)\widehat{U}(\xi,t), \\ \widehat{U}(\xi,0) = \widehat{U}_0(\xi) = (\widehat{u}_1^0(\xi), \widehat{u}_2^0(\xi), \widehat{v}_1^0(\xi), \widehat{v}_2^0(\xi), \widehat{w}^0(\xi), \widehat{y}^0(\xi), \theta^0), \end{cases} \tag{3.4.11}$$

where the time derivative is denoted by a prime, $\hat{U} = (\hat{u}, \hat{v}, \hat{w}, \hat{y}, \hat{\theta})^T = (\hat{u}_1, \hat{u}_2, \hat{v}_1, \hat{v}_2, \hat{w}, \hat{y}, \hat{\theta})^T$, and the matrix $\widehat{\mathcal{A}}$ is given by

$$\widehat{\mathcal{A}}(\xi) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ N_{31} & N_{32} & 0 & 0 & -iK\widetilde{\rho}^{-1}\xi_1 & 0 & -i\delta\widetilde{\rho}^{-1}\xi_1 \\ N_{41} & N_{42} & 0 & 0 & -iK\widetilde{\rho}^{-1}\xi_2 & 0 & -i\delta\widetilde{\rho}^{-1}\xi_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ iKJ^{-1}\xi_1 & iKJ^{-1}\xi_2 & 0 & 0 & -KJ^{-1}|\xi|^2 & 0 & \alpha \\ 0 & 0 & -i\delta\xi_1 & -i\delta\xi_2 & 0 & -\alpha & -|\xi|^2 \end{pmatrix}, \tag{3.4.12}$$

where

$$N_{31} = -\widetilde{\rho}^{-1}[D(\frac{1-\mu}{2}|\xi|^2 + \frac{1+\mu}{2}|\xi_1|^2) + K], \tag{3.4.13}$$

$$N_{32} = -\widetilde{\rho}^{-1}D(\frac{1+\mu}{2}\xi_1\xi_2), \tag{3.4.14}$$

$$N_{41} = -\widetilde{\rho}^{-1}D(\frac{1+\mu}{2}\xi_1\xi_2) \tag{3.4.15}$$

and

$$N_{42} = -\widetilde{\rho}^{-1}[D(\frac{1-\mu}{2}|\xi|^2 + \frac{1+\mu}{2}|\xi_2|^2) + K]. \tag{3.4.16}$$

Now, we show two eigenvalues of $\widehat{\mathcal{A}}$ which are purely imaginary. To this end, we make the following change of variables in polar coordinates

$$\xi_1 = r\cos\phi, \qquad \xi_2 = r\sin\phi,$$

and set :

$$\lambda_1(\xi) = i\frac{\sqrt{2K + D(1-\mu)|\xi|^2}}{\sqrt{2\widetilde{\rho}}} \in i\mathbb{R}, \tag{3.4.17}$$

$$\lambda_2(\xi) = -\lambda_1(\xi) \in i\mathbb{R}, \tag{3.4.18}$$

and consider the vectors.

$$\widehat{V}_1(\xi) = (\widehat{u}_{11}, \widehat{u}_{21}, \widehat{v}_{11}, \widehat{v}_{21}, \widehat{w}_1, \widehat{y}_1, \widehat{\theta}_1) = (-\sin\phi, \cos\phi, -\lambda_1(\xi)\sin\phi, \lambda_1(\xi)\cos\phi, 0, 0, 0), \tag{3.4.19}$$

$$\widehat{V}_2(\xi) = (\widehat{u}_{12}, \widehat{u}_{22}, \widehat{v}_{12}, \widehat{v}_{22}, \widehat{w}_2, \widehat{y}_2, \widehat{\theta}_2) = (-\sin\phi, \cos\phi, -\lambda_2(\xi)\sin\phi, \lambda_2(\xi)\cos\phi, 0, 0, 0). \tag{3.4.20}$$

Remark 3.4.2. Using a direct computation we prove that, for every $\xi \in \mathbb{R}^2$, $\lambda_1(\xi)$ and $\lambda_2(\xi)$ defined in (3.4.17)-(3.4.18) are two imaginary eigenvalues of $\widehat{\mathcal{A}}(\xi)$, where the associated eigenvectors are respectively $\widehat{V}_1(\xi)$ and $\widehat{V}_2(\xi)$.

Proof of Theorem 3.4.1. Let $\widehat{V}_1$ given by (3.4.19) (or similarly $\widehat{V}_2$ given by (3.4.20))) and set

$$\widehat{V}_0 : \xi \in \mathbb{R}^2 \mapsto \frac{1}{(1+|\xi|^2)^2}\widehat{V}_1(\xi).$$

Now, let $U_0 = \mathfrak{F}^{-1}(\widehat{V}_0)$.

First, we remark that $U_0 \in \mathcal{H}$. Indeed, from (3.4.17) and (3.4.19) we see that each component of $\widehat{V}_0$ are in $L^2(\mathbb{R}^2)$, hence each component of U_0 are in $L^2(\mathbb{R}^2)$, since the Fourier transform is an isometry from $L^2(\mathbb{R}^2)$ in $L^2(\mathbb{R}^2)$. Now, since $\mathfrak{F}(\partial_j u) = i\xi_j \widehat{u}, j = 1, 2$, then the norm of a function f in $H^1(\mathbb{R}^2)$ is given by:

$$\|f\|^2_{H^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2} (1 + |\xi|^2)|\widehat{f}(\xi)|^2 d\xi. \tag{3.4.21}$$

Therefore, it is easy to check that the two first components of U_0 are in $H^1(\mathbb{R}^2)$, consequently, U_0 is in $\mathcal{H}$.

Now, let $U(t)$ be the solution of (3.4.9) corresponding to the initial conditions U_0 given previously. Taking the Fourier transform of (3.4.9), then $\widehat{U}(t)$ is solution of

$$\partial_t \widehat{U}(t) = \widehat{\mathcal{A}}\, \widehat{U}(t), \text{ and } \widehat{U}(0) = \widehat{V}_0. \tag{3.4.22}$$

But, $\partial_t(e^{\lambda_1(\xi)t}\widehat{V}_0(\xi)) = \lambda_1(\xi)e^{\lambda_1(\xi)t}\widehat{V}_0(\xi) = \widehat{\mathcal{A}}(\xi)\left(e^{\lambda_1(\xi)t}\widehat{V}_0(\xi)\right)$, thus $e^{\lambda_1(\xi)t}\widehat{V}_0(\xi)$ is solution of (3.4.22). Since the solution is unique then $\widehat{U}(t) = e^{\lambda_1(\xi)t}\widehat{V}_0(\xi)$. Finally, by the fact that the Fourier transform is an isometry in $L^2(\mathbb{R}^2)$ and by (3.4.21) and since $\lambda_1(\xi) \in i\mathbb{R}$, we have

$$E(t) = \frac{1}{2}\|U(t)\|^2_{\mathcal{H}} = \frac{1}{2}\|U_0\|^2_{\mathcal{H}} = E(0).$$

The proof is thus complete.

3.5 Appendix

Lemma 3.5.1. *Let $\phi(r) = \min(1, r^2)$ and $c > 0$, then their exist $c_1 > 0$ and $c_2 > 0$ such that*

$$te^{-c\phi(r)t} \leq \frac{c_1}{\phi(r)} e^{-c_2\phi(r)t}, \forall t \geq 0.$$

Proof. To prove this lemma, it suffices to choose $c_2 = \dfrac{c}{2}$ and show the existence of a constant c_1 such that

$$\frac{c_1}{\phi(r)} \geq te^{-\frac{c}{2}\phi(r)t}, \forall t \geq 0.$$

Let

$$\varphi(t) = te^{-\frac{c}{2}\phi(r)t},$$

it is easy to see that $\varphi(t)$ has a maximum value at $t_0 = \dfrac{2}{c\phi(r)}$ and

$$\varphi(t_0) = \frac{2}{e\,c\,\phi(r)}.$$

Therefore it is sufficient to take

$$c_1 = \frac{2}{e\,c}.$$

□

Lemma 3.5.2. *Let H_1 (resp. H_2) be the solution of*

$$u''(t) + au'(t) + (\frac{K}{\rho} + \frac{Dr^2}{\rho})u(t) = 0, \quad t \geq 0, \tag{3.5.1}$$

which satisfies $H_1(0) = 1, H_1'(0) = 0$ (resp. $H_2(0) = 0, H_2'(0) = 1$), then there exists a constant $c > 0$ which does not depend on r such that the following estimates holds

$$|H_1(t)| \lesssim e^{-ct}, \ \forall t \geq 0, \tag{3.5.2}$$

$$|H_2(t)| \lesssim \frac{e^{-ct}}{\psi(r)} \textit{ with } \psi(r) = \max(1, r). \tag{3.5.3}$$

Proof. Let $\Delta = a^2 - 4(\frac{K}{\rho} + \frac{Dr^2}{\rho})$ and suppose that $\Delta < 0$ (i.e $a^2 - 4\frac{K}{\rho} \leq 0$ or that $a^2 - 4\frac{K}{\rho} > 0$ and $r > r_0 = \sqrt{\frac{\rho a^2}{4D} - \frac{K}{D}}$). Then

$$H_1(t) = e^{-\frac{a}{2}t}\left(\cos(\frac{\sqrt{-\Delta}}{2}t) + \frac{a}{2}t\frac{\sin(\frac{\sqrt{-\Delta}}{2}t)}{\frac{\sqrt{-\Delta}}{2t}}\right), \forall t > 0.$$

Therefore

$$|H_1(t)| \leq (1 + \frac{a}{2}te^{-\frac{a}{2}t}) \lesssim e^{-ct}.$$

If $\Delta = 0$ (i.e $a^2 - 4\frac{K}{\rho} > 0$ and $r = r_0$), then $H_1(t) = (1 + \frac{a}{2}t)e^{-\frac{a}{2}t}$, therefore the the previous estimate remains true.
Now if $\Delta > 0$ (i.e $a^2 - 4\frac{K}{\rho} > 0$ and $r < r_0$), then

$$H_1(t) = e^{-\frac{a}{2}t}\left(\cosh(\frac{t\sqrt{\Delta}}{2}) + \frac{a\,t}{2}\frac{\sinh(\frac{t\sqrt{\Delta}}{2})}{\frac{t\sqrt{\Delta}}{2}}\right).$$

For $t > 0$ fixed, using the fact that the function $u \in \mathbb{R}_+^* \mapsto \frac{\sinh(tu)}{tu}$ is increasing with respect to u we get

$$H_1(t) = |H_1(t)| \le e^{-\frac{a}{2}t}\, te^{t\sqrt{\frac{a^2}{4} - \frac{K}{\rho}}}(1 + \frac{a}{2\sqrt{\frac{a^2}{4} - \frac{K}{\rho}}}).$$

It follows that

$$|H_1(t)| \lesssim e^{(-\frac{a}{2} + \sqrt{\frac{a^2}{4} - \frac{K}{\rho}})t} \lesssim e^{-ct},$$

since $\frac{a}{2} - \sqrt{\frac{a^2}{4} - \frac{K}{\rho}} > 0.$
Now we give an estimate for $H_2(t)$. Assume that $a^2 - 4\frac{K}{\rho} < 0$, then $\Delta < 0$ and we find that

$$H_2(t) = 2e^{-\frac{a}{2}t}\frac{\sin(\frac{\sqrt{-\Delta}}{2}t)}{\sqrt{-\Delta}}.$$

It follows that

$$|H_2(t)| \lesssim \frac{e^{-\frac{a}{2}t}}{\psi(r)} \lesssim \frac{e^{-ct}}{\psi(r)}. \tag{3.5.4}$$

If $a^2 - 4\dfrac{K}{\rho} = 0$ then $\Delta < 0$ and

$$H_2(t) = e^{-\frac{a}{2}t} \frac{\sin \sqrt{\frac{D}{\rho}} rt}{\sqrt{\frac{D}{\rho}} r}.$$

It is easy to see that if $r \in]0, 1]$ then we have $|H_2(t)| \lesssim e^{-ct}$, whereas if $r \in]1, +\infty[$ then $H_2(t)| \lesssim \dfrac{e^{-ct}}{r}$. Therefore

$$|H_2(t)| \lesssim \frac{e^{-ct}}{\psi(r)}. \tag{3.5.5}$$

If $a^2 - 4\dfrac{K}{\rho} > 0$ and $r > r_0 = \sqrt{\dfrac{\rho a^2}{4D} - \dfrac{K}{D}}$ then $\Delta < 0$ and

$$H_2(t) = 2e^{-\frac{a}{2}t} \frac{\sin(\frac{\sqrt{-\Delta}}{2}t)}{\sqrt{-\Delta}}.$$

It follows that

$$|H_2(t)| \lesssim \frac{e^{-\frac{a}{2}t}}{\psi(r)} \lesssim \frac{e^{-ct}}{\psi(r)}. \tag{3.5.6}$$

Now if $a^2 - 4\dfrac{K}{\rho} > 0$ and $r < r_0$ then $\Delta > 0$ and

$$H_2(t) = te^{-\frac{a}{2}t} \frac{\sinh(\frac{\sqrt{\Delta}}{2}t)}{\frac{\sqrt{\Delta}}{2}t}.$$

Again, for $t > 0$ fixed, using the fact that the function $u \in \mathbb{R}_+^* \mapsto \dfrac{\sinh(tu)}{tu}$ is increasing with respect to u we get

$$|H_2(t)| \lesssim te^{(\frac{-a}{2} + \sqrt{\frac{a^2}{4} - \frac{K}{\rho}})t} \frac{1}{\sqrt{\frac{a^2}{4} - \frac{K}{\rho}}}$$

Therefore

$$|H_2(t)| \lesssim e^{-ct}. \tag{3.5.7}$$

Finally, from (3.5.4),(3.5.5),(3.5.6) and (3.5.7) we deduce the estimate (3.5.3)of $H_2(t)$.

□

Lemma 3.5.3. *Let K_1 (resp. K_2) be the solution of*

$$w''(t) + bw'(t) + \frac{Kr^2}{J} w(t) = 0, \quad t \geq 0, \tag{3.5.8}$$

which satisfies $K_1(0) = 1, K_1'(0) = 0$ (resp. $K_2(0) = 0, K_2'(0) = 1$), then there exists a constant $c > 0$ which does not depend on r such that the following estimates holds

$$|K_1(t)| \lesssim e^{-c\phi(r)t}, \ \forall t \geq 0, \tag{3.5.9}$$

$$|K_2(t)| \lesssim \frac{e^{-c\phi(r)t}}{\psi(r)}, \ \textit{with } \phi(r) = \min(1, r^2) \ \textit{ and } \psi(r) = \max(1, r). \tag{3.5.10}$$

Proof. We start by estimating $K_1(t)$.

Note that

$$\Re(\frac{-b - \sqrt{\Delta}}{2}) \leq -\frac{b}{2} < 0,$$

and there exists $c > 0$ such that

$$\Re(\frac{-b + \sqrt{\Delta}}{2}) \leq -c\phi(r) < 0. \tag{3.5.11}$$

where

$$\Delta = b^2 - \frac{4Kr^2}{J} \text{ and } \phi(r) = \min\{1, r^2\}.$$

Assume that $r > r_0 = \frac{b}{2}\sqrt{\frac{J}{K}}$ then $\Delta < 0$ and we find that

$$\begin{aligned} K_1(t) &= e^{\frac{-b}{2}t}\Big(\frac{bt}{2}\frac{\sin(\frac{\sqrt{-\Delta}}{2}t)}{\frac{\sqrt{-\Delta}}{2}t} + \cos(\frac{\sqrt{-\Delta}}{2}t)\Big) \\ &\leq (1+\frac{bt}{2})e^{\frac{-b}{2}t} \\ &\lesssim e^{-ct}. \end{aligned} \tag{3.5.12}$$

If $r = r_0$ then $\Delta = 0$ and we see that the previous estimate remains true.

Now if $r \in (\frac{r_0}{2}, r_0)$ then $\Delta < 0$ and

$$|K_1(t)| \leq b\,\frac{t}{2}\,e^{\frac{-b}{2}t}|\frac{\sinh(\frac{\sqrt{\Delta}}{2}t)}{\frac{\sqrt{\Delta}}{2}t}| + e^{\frac{-b}{2}t}|\cosh(\frac{\sqrt{\Delta}}{2}t)|.$$

For $t > 0$ fixed, using the fact that the function $u > 0 \mapsto \frac{\sinh(tu)}{tu}$ is increasing with respect to u we get

$$\begin{aligned} |K_1(t)| &\leq b\,e^{\frac{-b}{2}t}|\frac{\sinh(\frac{\sqrt{b^2-\frac{Kr_0^2}{J}}}{2}t)}{\sqrt{b^2-\frac{Kr_0^2}{J}}}| + e^{\frac{-b}{2}t}|\cosh(\frac{\sqrt{\Delta}}{2}t)| \;. \\ &\lesssim e^{-ct}. \end{aligned} \tag{3.5.13}$$

Now if $r \in (0, \frac{r_0}{2})$ then $\Delta < 0$ and

$$\begin{aligned} |K_1(t)| &\leq b\,e^{\frac{-b}{2}t}|\frac{\sinh(\frac{\sqrt{\Delta}}{2}t)}{\sqrt{\Delta}}| + e^{\frac{-b}{2}t}|\cosh(\frac{\sqrt{\Delta}}{2}t)| \\ &\lesssim e^{\frac{-b+\sqrt{\Delta}}{2}t}. \end{aligned}$$

using (3.5.11) in the last estimation, we get

$$|K_1(t)| \lesssim e^{-c\phi(r)t}. \tag{3.5.14}$$

Therefore, using (3.5.12),(3.5.13) and (3.5.14) we find the estimate (3.5.9) of $K_1(t)$.

Similarly, as we did to prove the estimate of $K_1(t)$ and the previous Lemma, we find

$$|K_2(t)| \lesssim \frac{e^{-c\phi(r)t}}{\psi(r)}.$$

□

Lemma 3.5.4. *The solutions λ of the equation $p_3(\lambda) = 0$ have negative real part .*

Proof. To prove Lemma 3.5.4, we will use Routh-Hurwitz criterion .
Substitute (**??**) and (**??**) in (3.3.37) to obtain

$$p_3(\lambda) = a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4.$$

where

$$a_0 = 1,\ a_1 = (a+b),\ a_2 = ab + \frac{K}{\rho} + \frac{Dr^2}{\rho} + \frac{Kr^2}{J},$$

and

$$a_3 = b(\frac{K}{\rho} + \frac{Dr^2}{\rho}) + a\frac{Kr^2}{J},\ a_4 = \frac{DKr^4}{\rho J}.$$

The Hurwitz matrix Form is given by

$$\begin{pmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{pmatrix},$$

The principal diagonal minors Δ_i of the Hurwitz matrix are given by the formulas

$$\Delta_1 = a_1, \quad \Delta_2 = \left| \begin{matrix} a_1 & a_0 \\ a_3 & a_2 \end{matrix} \right|, \quad \Delta_3 = \left| \begin{matrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{matrix} \right|, \text{ and } \Delta_4 = \left| \begin{matrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{matrix} \right|.$$

It is easy to prove that all the principal diagonal minors of the Hurwitz matrix are positive provided that $a_0 > 0$.
Therefore, the roots of $p_3(\lambda)$ have negative real parts . □

When it has a meaning we denote by $f * g$ the function defined for all $t \geq 0$.

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds, t \geq 0.$$

Lemma 3.5.5. *Assume that f, g are two functions such that $|f(t)| \lesssim e^{-c_1 t}$ and $|g(t)| \lesssim e^{-c_2 t}$, $t \geq 0$ then we have*

1. $|(f * g)(t)| \lesssim e^{-ct}$, *if if* $c_1, c_2 > 0$.
2. $|(f * g)(t)| \lesssim e^{-c\phi(r)t}$, *if* $c_1 > 0$, $c_2 = \phi(r)$.
3. $|(f * g)(t)| \lesssim \dfrac{1}{\phi(r)} e^{-c\phi(r)^2 t}$, *if* $c_1 = \tilde{c}_1 \phi(r)$ *and* $c_2 = \tilde{c}_2 \phi(r)^2$ $\tilde{c}_1, \tilde{c}_2 > 0$.

Proof. We only prove (3) since the remainder is easier and is left to the reader. First we remark that if $r > 1, \phi(r) = 1$, hence using (1) we only have to consider $r \leq 1$.

Without loss of generality we assume that $\tilde{c}_2 < \tilde{c}_1$, therefore we have for $r \leq 1$:

$$|(f * g)(t)| \leq \int_0^t e^{-\tilde{c}_1 r^2 (t-s)} e^{-\tilde{c}_2 r^4 s} ds = \frac{e^{-\tilde{c}_2 r^4 t} - e^{-\tilde{c}_1 r^2 t}}{r^2(\tilde{c}_1 - \tilde{c}_2 r^2)} \leq \frac{e^{-\tilde{c}_2 r^4 t}}{r^2(\tilde{c}_1 - \tilde{c}_2)},$$

and this last inequality shows (3). □

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Résumé

La thèse est portée essentiellement sur la stabilisation indirecte d'un système de deux équations des ondes couplées et sur la stabilisation frontière de poutre de Rayleigh.

Dans le cas de la stabilisation d'un système d'équations d'onde couplées, le contrôle est introduit dans le système directement sur le bord du domaine d'une seule équation dans le cas d'un domaine borné ou à l'interieur d'une seule équation mais dans le cas d'un domaine non borné. La nature du système ainsi couplé dépend du couplage des équations et de la nature arithmétique des vitesses de propagations, et ceci donne divers résultats pour la stabilisation polynômiale ainsi la non stabilité.

Dans le cas de la stabilisation de poutre de Rayleigh, l'équation est considerée avec un seul contrôle force agissant sur bord du domaine. D'abord, moyennant le développement asymptotique des valeurs propres et des vecteurs propres du système non contrôlé, un résultat d'oservabilité ainsi qu'un résultat de bornétude de la fonction de transfert correspendant sont obtenus. Alors, un taux de décroissance polynomial de l'énergie du système est établi. Ensuite, moyennant une étude spectrale combinée avec une méthode fréquentielle, l'optimalité du taux obtenu est assurée.

Mots clés : système de Timoshenko, Stabilité forte, Analyse spectrale, Stabilité polynomiale, Equation de la poutre de Rayleigh, fonction de transfert .

Study of the stability of a certain systems of coupled wave equations and of the Rayleigh beam equation on bounded and unbounded domains

Résumé en anglais

The thesis is driven mainly on indirect stabilization system of two coupled wave equations and the boundary stabilization of Rayleigh beam equation. In the case of stabilization of a coupled wave equations, the Control is introduced into the system directly on the edge of the field of a single equation in the case of a bounded domain or inside a single equation but in the case of an unbounded domain. The nature of thus coupled system depends on the coupling equations and arithmetic Nature of speeds of propagation, and this gives different results for the polynomial stability and the instability. In the case of stabilization of Rayleigh beam equation, we consider an equation with one control force acting on the edge of the area. First, using the asymptotic expansion of the eigenvalues and vectors of the uncontrolled system an observability result and a result of boundedness of the transfer function are obtained. Then a polynomial decay rate of the energy of the system is established. then through a spectral study combined with a frequency method, optimality of the rate obtained is assured.

Key words : Timoshenko system, Strong stability, Spectral analysis, Polynomial stability, Rayleigh beam equation, Transfert function .