Thése de doctorat

Pour obtenir le grade de Docteur de l'Université de

VALENCIENNES ET DU HAINAUT-CAMBRÉSIS

Discipline :

MATHÉMATIQUES APPLIQUÉES

Présentée et soutenue par Zainab, ABBAS.

Le 02/10/2014, VALENCIENNES

Ecole doctorale :
Sciences Pour l'Ingénieur (SPI)

Laboratoire :
Laboratoire de Mathématiques et ses Applications de Valenciennes (LAMAV)

Stabilisation de quelques équations d'évolution du second ordre par des lois de rétroaction

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Acknowledgements

Working on my PhD was an overwhelming experience, besides learning mathematics I encountered people whom I feel lucky to have met.

First and foremost, my sincere thanks go to my supervisor professor Serge Nicaise, not only for being an outstanding advisor and an excellent professor but also for being such a kind person whose spirit has been a source of encouragement for me all over the years of preparation of my thesis. From his enthusiasm and persistence on solving problems I learned that there's always a way, we just have to be patient enough to find it. For his optimism that gave me confidence throughout those years and for his help, I am grateful.

I am deeply indebted to my committee members Professor Bopeng Rao and Professor Roland Schnaubelt for their time and effort in reviewing this work. I would like also to thank Prof. Kais Ammari, Dr. Denis Mercier and Prof. Ali Wehbe for their participation as members of the jury and for their fruitful discussions regarding my work. I really appreciate your presence.

A big thanks go to the members of LAMAV of whom I mention Collete, Juliette, Denis, and Felix for their kindness and for helping me feel comfortable in working on my thesis. Not to forget mentioning Nabila Daifi and thank her for the administrative help and her good spirit.

I am grateful to my professors at the Lebanese University, for whom is the credit of my background in mathematics.

A big thank you to my best friend Sadjia, thank you for being my friend and thank you for supporting and encouraging me. I would like to express my gratitude towards my friends Cindy, Linda and Farah, we shared a lot of memories that I won't forget. I would like also to thank Asmaa, Sara, Yomna, Zeineb, Ihsen, Anis, Kassem, Hossep, Yamen, Basma, Amina and all my friends outside the University.

For my dad's words of wisdom, the love of my mom, the advices of my brother, the kind words of my sister, for all this and more I am forever grateful.

Finally, thanks for every kind word and nice wish.

Abstract

In this thesis, we study the stabilization of some evolution equations by feedback laws. In the first chapter we study the wave equation in $\mathbb{R}$ with dynamical boundary control applied on a part of the boundary and a Dirichlet boundary condition on the remaining part. We furnish sufficient conditions that guarantee a polynomial stability proved using a method that combines an observability inequality for the associated undamped problem with regularity results of the solution of the undamped problem. In addition, the optimality of the decay is shown in some cases with the help of precise spectral results of the operator associated with the damped problem. Then in the second chapter we consider the system on a domain of $\mathbb{R}^d$, $d \geq 2$. In this case, the domain of the associated operator is not compactly embedded into the energy space. Nevertheless, we find sufficient conditions that give the strong stability. Then, we discuss the non uniform stability as well as the polynomial stability by two methods. The frequency domain approach allows us to establish a polynomial decay on some domains for which the wave equation with the standard damping is exponentially or polynomially stable. Finally, in the third chapter we consider a general framework of second order evolution equations with dynamical feedbacks. Under a regularity assumption we show that observability properties for the undamped problem imply decay estimates for the damped problem. We finally illustrate our general results by a variety of examples.

Keywords. Acoustics, stability, evolution equations, observability, Riesz basis, wave equation.

Résumé

Dans cette thèse, nous étudions la stabilisation de certaines équations d'évolution par des lois de rétroaction. Dans le premier chapitre nous étudions l'équation des ondes dans $\mathbb{R}$ avec conditions aux limites dynamiques appliquées sur une partie du bord et une condition de Dirichlet sur la partie restante. Nous fournissons des conditions suffisantes qui garantissent une stabilité polynomiale en utilisant une méthode qui combine une inégalité d'observabilité pour le problème non amorti associé avec des résultats de régularité du problème non amorti. L'optimalité de la décroissance est montrée dans certains cas à l'aide des résultats spectraux précis de l'opérateur associé. Dans le deuxième chapitre nous considérons le système sur un domaine de $\mathbb{R}^d$, $d \geq 2$. On trouve des conditions suffisantes qui permettent la stabilité forte. Ensuite, nous discutons de la stabilité non uniforme ainsi que de la stabilité polynomiale. L'approche en domaine fréquentiel nous permet d'établir une décroissance polynomiale sur des domaines pour lesquels l'équation des ondes avec l'amortissement standard est exponentiellement ou polynomialement stable. Dans le troisième chapitre nous considérons un cadre général d 'équations d'évolution avec une dissipation dynamique. Sous une hypothèse de régularité, nous montrons que les propriétés d'observabilité pour le problème non amorti impliquent des estimations de décroissance pour le problème amorti.

Mots-clés. Stabilité, équations d'évolution, observabilité, base de Riesz, équation des ondes.

Contents

Introduction

Control theory is the study of the process of controlling the behavior of an operator system to achieve a certain target. Its application ranges widely from earthquake engineering and seismology to fluid transfer, cooling water and noise reduction in cavities, vehicles, such as pipe systems. Acoustics, aeronautics, hydraulics, are also some of the diverse disciplines where control theory is applied. Of the most important notions in modern systems and control theory we mention controllability, stabilizability and observability. Various types of those notions have been introduced for abstract systems defined on Banach or Hilbert spaces and the relations between them has been extensively explored by several authors.

Roughly speaking, the concept of controllability is defined as an ability to do whatever we want with our system. In more technical terms, it is described as the ability to steer our evolution system, whether described in terms of partial or ordinary differential equations, from any initial state to any desired final state in a finite time interval by means of a suitable control (boundary control, internal control, controls localized on open subsets of bounded sets, etc...). The definition of the same concept varies according to the framework or the type of models on which it is applied. The three types of controllability that are mainly defined are exact controllability, null controllability and asymptotic controllability (see [24]). The differences between those definitions are examined for both finite dimensional systems and infinite dimensional systems for time reversible systems (e.g. wave equation) as well as time irreversible systems (e.g. heat equation). In general the different types of controllability are not equivalent. The relations between these concepts were studied and several results on this subject were obtained (see for instance the works of Micu and Zuazua [35]).

Observability is a measure for how well internal states of a system can be inferred by knowledge of its external outputs. The duality between the controllability and observability of systems of partial differential equations in Banach spaces has been examined in many works such as those of Lions [30] where Hilbert uniqueness theorem HUM is explained (see also [39]), and the works of Russell and Dolecki and Russell [22, 45, 47]. Various methods could be used to prove observability inequalities such as Carleman estimates, microlocal analysis and the multiplier method. For more details on the treatment of observability problems and proving observability inequalities for linear systems, we refer the reader to [55], [52], and [30].

As for stabilizability, it is defined as the ability to find an input control that requires the state response to approache zero as time $t \to \infty$. Different types of stability also occurs. The details of the notions of stability used in our thesis are explained below.

In order to introduce the main theme of our study, the used method and the obtained results let us recall some of the fundamental definitions that are being used throughout the thesis.

Definition 0.1 *Let X be a Banach space. A one parameter family $(S(t))_{t\geq 0}$ of bounded linear operators defined from X into X is a strongly continuous semigroup of bounded linear operators on X if:*

- $S(0) = I$,*(I identity operator on X).*
- $S(t+s) = S(t)S(s)$ *for every* $t, s \geq 0$.
- $S(t)x \to x$, *as* $t \to \infty$, $\forall x \in X$.

Such a semigroup is called a C_0-semigroup.

Definition 0.2 *The infinitesimal generator $\mathcal{A}$ of the semigroup $(S(t))_{t\geq 0}$ is defined by:*

$$D(\mathcal{A}) = \{x \in X | \lim_{t\to\infty} \frac{S(t)x - x}{t} exists\}$$

and

$$\mathcal{A}x = \lim_{t\to\infty} \frac{S(t)x - x}{t}, \ x \in D(\mathcal{A}).$$

Definition 0.3 *Let $\mathcal{H}$ be a Hilbert space. An operator $(\mathcal{A}, D(\mathcal{A}))$ on $\mathcal{H}$ satisfying*

$$\Re(\mathcal{A}u, u) \leq 0, \forall u \in D(\mathcal{A}),$$

is said to be a dissipative operator. A maximal dissipative operator $(\mathcal{A}, D(\mathcal{A}))$ on $\mathcal{H}$ is a dissipative operator for which $R(\lambda I - \mathcal{A}) = \mathcal{H}$, for some $\lambda > 0$. A maximal dissipative operator is also called m-dissipative operator.

Generally speaking, the first step in dealing with the study of the stability of the solution is to rewrite our evolution system of partial differential equations as a Cauchy problem on some appropriate Hilbert space $\mathcal{H}$ called the energy space

$$\dot{U} = \mathcal{A}U, \quad U(0) = U_0. \tag{1}$$

where $\mathcal{A}$ is an unbounded operator on $\mathcal{H}$. Then we prove that $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions $(S(t))_{t\geq 0}$ on $\mathcal{H}$ in order to deduce the existence of a solution in a certain Hilbert space. The solution is hence of the form $U(t) = S(t)U_0$. We mention here Lumer-Phillips theorem (see [32]) which is applied to justify the existence and uniqueness of solutions of some partial differential equations.

Theorem 0.4 (Lumer-Phillips theorem) *Let $\mathcal{A}$ be a linear operator with dense domain $D(\mathcal{A})$ in a Banach space X.*

(a) If $\mathcal{A}$ is dissipative and there exists $\lambda_0 > 0$ such that $R(\lambda_0 I - \mathcal{A}) = X$ then $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on X.

(b) If $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on X then $R(\lambda I - \mathcal{A}) = X$ for all $\lambda > 0$ and $\mathcal{A}$ is dissipative.

Consequently, $\mathcal{A}$ is maximal dissipative on a Hilbert space $\mathcal{H}$ if and only if it generates a C_0-semigroup of contractions on $\mathcal{H}$ and thus the existence of the solution is justified by the following corollary which follows from Lumer-Phillips theorem.

Corollary 0.5 *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{A}$ be a linear operator defined from $D(\mathcal{A}) \subset \mathcal{H}$ into $\mathcal{H}$. If $\mathcal{A}$ is maximal dissipative then the initial value problem*

$$\begin{cases} \frac{du}{dt}(t) & = & \mathcal{A}u(t), \ t > 0 \\ u(0) & = & u_0 \end{cases}$$

has a unique solution $u \in C([0,+\infty),\mathcal{H})$, for each initial datum $u_0 \in \mathcal{H}$. Moreover, if $u_0 \in D(\mathcal{A})$, then

$$u \in C([0,+\infty), D(\mathcal{A})) \cap C^1([0,+\infty),\mathcal{H}).$$

After proving the well posedness of the systems introduced in the chapters of the thesis, we aim to discuss the type of stability of the solution of the systems formulated as (1). We introduce here the notions of stability that we encounter in this work (see [15] and [14] for instance).

Assume that $\mathcal{A}$ is a generator of a strongly continuous semigroup of contractions on a Hilbert space $\mathcal{H}$. We say that the semigroup $(S(t))_{t\geq 0}$ generated by $\mathcal{A}$ is

- Strongly (asymptotically) stable if for all $U_0 \in \mathcal{H}$

$$\|S(t)U_0\|_{\mathcal{H}} \to 0.$$

- Exponentially stable if there exist two positive constants C, ω such that

$$\|S(t)U_0\|_{\mathcal{H}} \leq Ce^{-\omega t}\|U_0\|_{\mathcal{H}}, \forall t > 0, \forall U_0 \in \mathcal{H}.$$

- Polynomially stable if there exist constants $\alpha, \beta, C > 0$ such that

$$\|S(t)(d - \mathcal{A})^{-\alpha}\| \leq Ct^{-\beta}, \ t > 0,$$

  for some $d > 0$.

Clearly the definitions of the different kinds of stability could be introduced for the energy of the solution of (1) defined by $E(t) = \frac{1}{2}\|U(t)\|^2$.

In order to show the strong stability of $U(t) = S(t)U_0$ we study the spectrum of the operator $\mathcal{A}$ of (1) and we show that the only pure imaginary elements of its spectrum are countable and belong exclusively to its essential spectrum. The asymptotic stability is thus deduced by Arendt-Batty Theorem (see [10] and Theorem 1.3.1 in the first chapter). The discussion of the type of stability achieved by our systems is detailed later and is based in the first chapter on the analysis of the spectrum of the operator of a conservative operator associated with the dissipative operator in order to obtain an observability inequality using Ingham's inequality (see [11]). While in the second chapter we introduce, as in [38], a Lyapunov functional or a resolvent method from [19] (see also [15]) to find appropriate estimates on the energy. We moreover use Huang-Prüss Theorem (see [23, 26, 43]) and a result of [44] (see also [50]) to prove the nonuniform stability in certain cases.

Let us now briefly explain the contents of our thesis. This thesis is divided into three chapters. The aim of the first and second chapter is to study the stability of the wave equation defined on an open connected bounded set Ω of $\mathbb{R}^d$, $d \geq 1$ with a boundary $\partial\Omega = \Gamma$ assumed to be divided into two disjoint parts Γ_0 and Γ_1 , where Γ_0 is assumed to be closed with a nonempty interior and

Γ_1 relatively open in Γ which could be possibly empty for $d > 1$. A clamped boundary condition is assumed to be satisfied on a part Γ_1 of the boundary and a dynamic boundary condition on the second part Γ_0. More precisely, we consider the system defined by

$$\begin{cases} y_{tt}(x,t) - \Delta y(x,t) = 0 & , x \in \Omega, t > 0, \\ y(x,t) = 0 & , x \in \Gamma_1, t > 0, \\ \frac{\partial y}{\partial \nu}(x,t) = (\delta(x,t), C) & , x \in \Gamma_0, t > 0, \\ \delta_t(x,t) = B\delta(x,t) - Cy_t(x,t) & , x \in \Gamma_0, t > 0, \end{cases} \tag{2}$$

with the following initial conditions:

$$\begin{cases} y(x,0) = y_0(x), y_t(x,0) = y_1(x), \quad x \in \Omega, \\ \delta(x,0) = \delta_0(x), x \in \Gamma_0, \end{cases} \tag{3}$$

where $(\cdot,\cdot)$ an inner product on $\mathbb{C}^n$ with $n \geq 1$, $B \in M_n(\mathbb{C})$, $C \in \mathbb{C}^n$ if $d = 1$ and $C \in C^{0,1}(\Gamma_0, \mathbb{C}^n)$ and $B \in C(\Gamma_0, M_n(\mathbb{C}))$ if $d > 1$.

The damping of the system is made via the indirect damping mechanism on the part Γ_0 that involves a first order differential equation in the variable δ. The notion of indirect damping mechanisms has been introduced by Russell in [49] and retains the attention of many authors (see for instance [5, 6, 48]). In addition, different models from acoustic theory enter in this framework.

The case $n = 1$ was considered in [53] ($d = 1$) and in [50] ($d \geq 2$) where a polynomial decay in $\frac{1}{t}$ was proved for initial data in the domain of the associated operator by using the multiplier method, leading to strong geometrical assumption on Γ_0. In the case $n \geq 2$, the third and fourth equations in (2) are general versions of the so-called acoustic boundary conditions, introduced for $n = 2$ in [17]. Acoustic boundary conditions arise in many physical applications, in particular they occur in theoretical acoustics, where a part of the boundary is not rigid but subject to small oscillations, see [16, 18, 31, 37, 38] and the references therein for more details. Absorbing boundary conditions like in [13] are stronger and lead to exponential decay of the energy, but they do not enter into our framework.

The stability of the wave equation with acoustic boundary condition was first studied by Beale in [16] where he discusses the strong stability of the system,

$$\begin{cases} \phi_{tt}(x,t) - c^2\Delta\phi(x,t) & = & 0, & x \in \Omega, t > 0 \\ \eta_t(x,t) - \frac{\partial \phi}{\partial \nu}(x,t) & = & 0, & x \in \Gamma_0, t > 0 \\ m(x)\eta_{tt}(x,t) + d(x)\eta_t(x,t) + k(x)\eta(x,t) + \rho\phi_t(x,t) & = & 0, & x \in \Gamma_0 \quad t > 0 \end{cases} \tag{4}$$

with m, d, k are positive sufficiently smooth functions defined on Γ_0 . In this case, $\Gamma_1 = \emptyset$ and system (4) can be formulated for $c = 1$ as (2)by taking

$$n = 2, \delta = \begin{pmatrix} \eta \\ \eta_t \end{pmatrix}, \; B = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{pmatrix}, \quad C = \begin{pmatrix} 0 \\ \frac{\rho}{m} \end{pmatrix}.$$

The third condition of (4) is the acoustic boundary condition introduced in [17]. In [38], Rivera and Qin studied the stability of (2) on $\Omega \subset \mathbb{R}^3$ with boundary

$$\partial\Omega = \Gamma_0 \cup \Gamma_1, \bar{\Gamma_0} \cap \bar{\Gamma_1} = \emptyset, \text{ and } \text{ meas } \Gamma_1 \neq 0,$$

assuming moreover the existence of a point $x_0 \in \mathbb{R}^3$ such that

$$\Gamma_1 = \{x \in \Gamma | (x - x_0).\nu \leq 0\}, \ \Gamma_0 = \{x \in \Gamma | (x - x_0).\nu \geq a > 0\},$$

for some constant $a > 0$. By introducing an appropriate Lyapunov function, the authors prove that the energy decays polynomially with a decay rate of $\frac{1}{t}$. In [31], the authors consider the wave equation on $\Omega \subset \mathbb{R}^2$ with acoustic boundary condition on one part of the boundary but replacing the Dirichlet boundary condition on Γ_1 by a Neumann boundary condition. Again the authors obtain a polynomial decay rate depending on the regularity of the initial data.

In the first chapter we study the stability of (2) for $\Omega = (0, 1)$ with $\Gamma_1 = 0, \Gamma_0 = 1$. Then, system (2) is given by

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) & = & 0, & 0 < x < 1, t > 0, \\ y(0,t) & = & 0, & t > 0, \\ y_x(1,t) - (\delta(t), C)_{\mathbb{C}^n} & = & 0, & t > 0, \\ \delta_t(t) - B\delta(t) + Cy_t(1,t) & = & 0, & t > 0, \end{cases} \tag{5}$$

Using the compact perturbation result of Russell [46], the dissipative system (5) is not uniformly stable (see section 1.4.1 of the first chapter and [37, Rk 2]). Hence we are interested in proving a weaker decay of the energy. More precisely we will give sufficient conditions on B and C that yield the polynomial decay of the energy of our system (for initial data in the domain of the associated operator). Contrary to [53] and [38] where a multiplier method is used, we here use a technique, inspired from [8, 34, 41], that consists of combining an observability inequality for the associated undamped problem obtained via sharp spectral results with regularity results of the solution of the undamped problem with a specific right-hand side. Moreover, using a careful spectral analysis of the operator associated with (5) we show in some particular situations that our decay rate is optimal.

In the second chapter we study the stability of (2) for $\Omega \subset \mathbb{R}^d$ with $d > 1$. We further assume that the boundary $\partial\Omega = \Gamma$ is Lipschitz and that that $\Gamma_0 \cap \bar{\Gamma}_1$ is of class C^1 in the sense explained in the second chapter. In a first step we try to find sufficient conditions that guarantee the strong stability of the system. Here as the domain of the associated operator is not compactly embedded into the natural energy space, we can expect that its spectrum is not only made up of eigenvalues. We prove such a result in our general setting but since Dirichlet boundary conditions are imposed on a part of the boundary, we were not able to use the single-layer potential technique of [16] and instead we use a Fredholm alternative technique. Finally, similar assumptions on B and C as in the one-dimensional case allow us to show that the associated operator has no eigenvalues on the imaginary axis, hence we can obtain the strong stability by using Arendt-Batty theorem (see [10] and Theorem 1.3.1). In dimension larger than 2, we can not apply the compact perturbation result of Russell [46] (see also section 1.4.1 of the first chapter) in order to prove the non uniform stability of (2). Nevertheless by using the spectral properties of the Laplace operator with specific Robin boundary conditions on Γ_0, we will show that the resolvent of the associated operator is not uniformly bounded on the imaginary axis and by the frequency domain approach [23, 26, 43], we will conclude that our system is not uniformly stable. Hence we are interested in proving a weaker decay of the energy. More precisely we will give sufficient conditions on Γ_0, B and C that yield the polynomial decay of the energy of our system (for initial data in the domain of the associated operator). A first approach is to use a multiplier method like in [38, 50, 53] but this approach requires a quite strong geometrical assumption on Γ_0. Hence we alternatively use the frequency domain approach from [19]. In that case, we prove some appropriate bound for the resolvent on the imaginary axis by using the exponential or polynomial decay of the wave equation with the standard damping

$$\frac{\partial y}{\partial \nu}(x,t) = -y_t \text{ on } \Gamma_0,$$

and an assumption on the behavior of $\Re((isI-B)^{-1}C, C)$ for all real number s with modulus large enough. This leads to quite weaker geometrical assumption on Γ_0 due to the results from [12, §5] or [28, 29] for instance. In particular in this second approach as Γ_1 can be empty, we significantly improve results from [16] and [38].

In the third chapter, we study the stability of linear control problems coming from elasticity which can be written as

$$\begin{cases} x''(t) + Ax(t) + Bu(t) = 0, & t \in [0, +\infty) \\ u'(t) - \widehat{C}u(t) - B^*x'(t) = 0, & t \in [0, +\infty) \\ x(0) = x_0, x'(0) = y_0, u(0) = u_0, & \end{cases} \tag{6}$$

where X and U are two complex Hilbert spaces, $x : [0, +\infty) \to X$ is the state of the system, $u \in L^2(0, T; U)$ is the input function, A is an unbounded positive self-adjoint operator on X, $B \in \mathcal{L}(U, \mathcal{D}(A^{\frac{1}{2}})')$ and $\widehat{C}$ is a maximal dissipative operator on U. The second equation of the considered system describes a dynamic control in some models. Some systems that can be covered by the formulation (6) are for example the hybrid systems. System (2) considered in the first chapter can be viewed as an application as well.

In this chapter we give sufficient conditions leading to the uniform or non uniform stability of the solutions of the corresponding closed loop system. We first justify the well-posedness of the problem then we write $\widehat{C}$ as a sum of a skew-adjoint operator $-C$ and a self-adjoint operator $-DD^*$ and we prove under a regularity assumption that the observability properties, described by assumption (O), of the undamped problem corresponding to replacing $\hat{C}$ by $-C$ in (6) imply decay estimates for the damped problem. We present in the last section of this chapter illustrative examples as applications of the general setting where we obtain polynomial or exponential energy decay rates. Finally, we note that we use a variety of methods in verifying the observability assumption as well as a regularity assumption when $\hat{C}$ is unbounded.

Note that the chapters of this thesis correspond to articles which have been published [2] or submitted [1, 3, 4]. Thus we have kept the general structure of the articles but just regrouping [3] and [4] in one chapter.

Let us finish this introduction with some notation used in the remainder of the thesis: the notation $A \lesssim B$ and $A \sim B$ means the existence of positive constants C_1 and C_2, which are independent of A and B such that $A \leq C_2 B$ and $C_1 B \leq A \leq C_2 B$.

1 Polynomial decay rate for a wave equation with general acoustic boundary feedback laws

1.1 Introduction

We consider the following one-dimensional evolution problem with a Dirichlet boundary condition at one end and a dynamical control at the other one, described as follows:

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) &= 0, \quad 0 < x < 1, t > 0, \\ y(0,t) &= 0, \quad t > 0, \\ y_x(1,t) + (\eta(t), C)_{\mathbb{C}^n} &= 0, \quad t > 0, \\ \eta_t(t) - B\eta(t) - Cy_t(1,t) &= 0, \quad t > 0, \end{cases} \tag{1.1}$$

with the following initial conditions:

$$\begin{cases} y(x,0) = y_0(x), y_t(x,0) = y_1(x), \quad 0 < x < 1, \\ \eta(0) = \eta_0, \end{cases} \tag{1.2}$$

where $B \in M_n(\mathbb{C})$, $C \in \mathbb{C}^n$ are given, y represents the transverse displacement of the vibrating string and η denotes the dynamical control variable. Define

$$V = \{y \in H^1(0,1) : y(0) = 0\},$$

endowed with the following inner product $(y,z)_V = \int_0^1 y_x \bar{z}_x dx$, and the energy space

$$\mathcal{H} = V \times L^2(0,1) \times \mathbb{C}^n,$$

endowed with the following inner product,

$$((y,z,\eta),(y_1,z_1,\eta_1))_{\mathcal{H}} = \int_0^1 y_x \bar{y}_{1x} dx + \int_0^1 z\bar{z}_1 dx + (\eta,\eta_1)_{\mathbb{C}^n},$$

with $(\cdot,\cdot)_{\mathbb{C}^n}$ an inner product on $\mathbb{C}^n$ to be well determined. Denote by M the Hermitian positive definite matrix associated with this inner product. For shortness we sometimes use the notation $(\cdot,\cdot)$ to denote $(\cdot,\cdot)_{\mathbb{C}^n}$ throughout the rest of the work.

The damping of the system is made via the indirect damping mechanism at the extremity 1 that involves a first order differential equation in η. Using the compact perturbation result of Russell [46], the dissipative system (1.1) is not uniformly stable (see section 1.4.1 and [37, Rk 2]). Hence we are interested in proving a weaker decay of the energy. More precisely we will give sufficient conditions on B and C that yield the polynomial decay of the energy of our system (for initial data in the domain of the associated operator). Contrary to [53] and [38] where a multiplier method is used, we here use a technique, inspired from [8, 34, 41], that consists of combining an observability inequality for the associated undamped problem obtained via sharp spectral results with regularity results of the solution of the undamped problem with a specific right-hand side. Moreover, using a careful spectral analysis of the operator associated with (1.1) we show in some particular situations that our decay rate is optimal.

The chapter is organized as follows. The second section deals with the well-posedness of the problem obtained by using semigroup theory. Section 1.3 is devoted to the analysis of the strong stability of the system. In section 1.4, we perform the spectral analysis of the operator associated with the conservative system and deduce the polynomial stability of the dissipative system. The optimality of the decay is considered in section 1.5. Finally some particular examples illustrating our general framework are presented in section 1.6.

1.2 Well-posedness results

In order to solve system (1.1) we use a reduction order argument. Define the linear operator $\mathcal{A}$ by

$$D(\mathcal{A}) = \{(y, z, \eta) \in H^2(0,1) \cap V \times V \times \mathbb{C}^n : y_x(1) = -(\eta, C)_{\mathbb{C}^n}\},$$

and

$$\mathcal{A}\begin{pmatrix} y \\ z \\ \eta \end{pmatrix} = \begin{pmatrix} z \\ y_{xx} \\ B\eta + Cz(1) \end{pmatrix}, \forall \begin{pmatrix} y \\ z \\ \eta \end{pmatrix} \in D(\mathcal{A}).$$

We reformulate our problem into a Cauchy problem given by

$$\dot{u} = \mathcal{A}u, \quad u(0) = u_0 \tag{1.3}$$

with $u = (y, z, \eta)^T$ and $u_0 = (y_0, y_1, \eta_0)^T$. We proceed by proving that $\mathcal{A}$ is m-dissipative. The existence of a unique solution of problem (1.3) follows from Lumer-Phillips Theorem (see for instance [42]).

Proposition 1.2.1 *Suppose that*

$$\Re(B\eta, \eta)_{\mathbb{C}^n} \le 0, \forall \eta \in \mathbb{C}^n. \tag{1.4}$$

Then the operator $\mathcal{A}$ is m-dissipative, thus $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$.

Proof. Let $\begin{pmatrix} y \\ z \\ \eta \end{pmatrix} \in D(\mathcal{A})$, we have:

$$\begin{aligned}\left(\mathcal{A}\begin{pmatrix} y \\ z \\ \eta \end{pmatrix}, \begin{pmatrix} y \\ z \\ \eta \end{pmatrix}\right)_{\mathcal{H}} &= \int_0^1 z_x \bar{y}_x dx + \int_0^1 y_{xx}\bar{z} dx + (B\eta + Cz(1), \eta)_{\mathbb{C}^n} \\ &= 2i\Im\left(\int_0^1 z_x \bar{y}_x dx + z(1)(C, \eta)_{\mathbb{C}^n}\right) + (B\eta, \eta)_{\mathbb{C}^n}.\end{aligned}$$

Hence $\Re(\mathcal{A}U,U) = \Re(B\eta,\eta)_{\mathbb{C}^n} \leq 0$, and thus $\mathcal{A}$ is dissipative.
We would like to show that there exists $\lambda > 0$ such that $\lambda I - \mathcal{A}$ is surjective. Let $\lambda > 0$ be given. Clearly, we have $\lambda \notin \sigma(B)$. For $(y_1, z_1, \eta_1) \in \mathcal{H}$, we look for $(y, z, \eta) \in D(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A}) \begin{pmatrix} y \\ z \\ \eta \end{pmatrix} = \begin{pmatrix} y_1 \\ z_1 \\ \eta_1 \end{pmatrix},$$

i.e. we are searching for $y \in H^2 \cap V, \eta \in \mathbb{C}^n$ satisfying

$$y_x(1) = -(\eta, C), \eta = (\lambda I - B)^{-1} \left(\eta_1 + C\left(\lambda y(1) - y_1(1)\right)\right),$$

and the following strong problem:

$$\lambda^2 y - y_{xx} = z_1 + \lambda y_1 = f_1.$$

We now define the associated weak problem and we then prove that it admits a unique solution using Lax-Milgram lemma. We state the weak problem as follows, find $y \in V$ satisfying

$$a(y,\varphi) = L(\varphi), \quad \forall \varphi \in V \tag{1.5}$$

where the conjugates of a and L are given by

$$\bar{a}(y,\varphi) = \int_0^1 \lambda^2 y\bar{\varphi}dx + \int_0^1 y_x\bar{\varphi}_x dx + ((\lambda I - B)^{-1}C, C)_{\mathbb{C}^n}\lambda y(1)\bar{\varphi}(1),$$

$$\bar{L}(\varphi) = \int_0^1 f_1\bar{\varphi}dx + \left((\lambda I - B)^{-1}(Cy_1(1) - \eta_1), C\right)_{\mathbb{C}^n}\bar{\varphi}(1).$$

Clearly, V is a Hilbert space, L is a linear continuous functional defined on V and a is a sesquilinear continuous form on V. Finally, a is coercive since $|a(y,y)| \geq \Re a(y,y)$. Indeed,

$$a(y,y) = \lambda^2 \int_0^1 |y|^2 dx + \int_0^1 |y_x|^2 dx + ((\lambda I - B)^{-1}C, C)\lambda|y(1)|^2,$$

and $\Re(B\eta,\eta) \leq 0$ implies $\Re((\lambda I - B)^{-1}C, C) \geq 0$, since

$$\begin{aligned} \Re\left((\lambda I - B)^{-1}C, C\right) &= \Re\left(u, (\lambda I - B)u\right) \\ &= \lambda\|u\|^2 - \Re(u, Bu) \geq 0, \end{aligned}$$

with $u = (\lambda I - B)^{-1}C$. Hence $\Re a(y,y) \gtrsim \|y\|_V^2$, which implies that a is coercive.
Applying Lax-Milgram Lemma, there exists a unique $y \in V$ solution of equation (1.5).
In particular, setting $\varphi \in \mathcal{D}(0,1)$ in (1.5), we get

$$\lambda^2 y - y_{xx} = f_1, \text{ in } \mathcal{D}'(0,1). \tag{1.6}$$

Due to the fact that $y \in V$ we get $y_{xx} \in L^2(0,1)$, and we deduce that $y \in H^2(0,1)$. Multiplying both sides of the conjugate of equality (1.6) by a $\phi \in V$, integrating by parts on $(0,1)$, and comparing with (1.5) we get

$$y_x(1) = \left((\lambda I - B)^{-1}\left(Cy_1(1) - C\lambda y(1) - \eta_1\right), C\right).$$

Defining $\eta = (\lambda I - B)^{-1}(\eta_1 - Cy_1(1) + C\lambda y(1))$, we get $y_x(1) = -(\eta, C)$ and by choosing $z = \lambda y - y_1$ we deduce the surjectivity of $\lambda I - \mathcal{A}$. Finally, we conclude that $\lambda I - \mathcal{A}$ is bijective, for all $\lambda > 0$. ■

Remark 1.2.2 *If* 0 *is not an eigenvalue of* B, *then* $\mathcal{A}$ *is bijective (see Proposition 1.3.2 below) and* $\mathcal{A}^{-1}$ *is bounded.*

Remark 1.2.3 *If* $\Re(B\eta,\eta) \leq 0$, *then* $\Re\left((izI-B)^{-1}C, C\right) \geq 0$. *Indeed,*

$$\begin{aligned} \Re((izI-B)^{-1}C, C) &= \Re(u, (izI-B)u) = \Re(u, izu) - \Re(u, Bu) \\ &= \Re(-iz\|u\|^2) - \Re(u, Bu) = -\Re(u, Bu) \geq 0. \end{aligned}$$

Since $\mathcal{A}$ is m-dissipative, then Lumer-Phillips theorem allows us to state the following corollary.

Corollary 1.2.4 *(i) For an initial datum* $u_0 \in \mathcal{H}$ *there exists a unique solution* $u \in C([0,+\infty), \mathcal{H})$ *to problem (1.3). Moreover, if* $u_0 \in D(\mathcal{A})$, *then*

$$u \in C([0,+\infty), D(\mathcal{A})) \cap C^1([0,+\infty), \mathcal{H}).$$

(ii) For each $u_0 \in D(\mathcal{A})$, *the energy* $E(t)$ *of the solution* u *of problem (1.3), defined by*

$$E(t) = \frac{1}{2}\|u(t)\|^2_{\mathcal{H}},$$

satisfies

$$\frac{d}{dt}E(t) = \Re(B\eta, \eta),$$

therefore the energy is non-increasing.

Proof. (i) is a direct consequence of Lumer-Phillips theorem.
(ii) holds simply since

$$\frac{dE(t)}{dt} = \Re(\frac{du(t)}{dt}, u(t)) = \Re(\mathcal{A}u, u).$$

■

1.3 Asymptotic stability

Since $\mathcal{A}$ is m-dissipative and $D(\mathcal{A})$ is compactly embedded in $\mathcal{H}$, then for all $\lambda > 0$ the operator $(\lambda I - \mathcal{A})^{-1}$ is compact. Thus $\mathcal{A}$ has a compact resolvent, which implies that the spectrum of $\sigma(\mathcal{A})$ is equal to its discrete spectrum $\sigma_d(\mathcal{A})$. To show that $(T(t))_{t>0}$ generated by $\mathcal{A}$ is stable we are going to use the following theorem due to Arendt and Batty (see [10]).

Theorem 1.3.1 (Arendt-Batty) *Let* X *be a reflexive Banach space. Assume that* T *is bounded and no eigenvalues of* $\mathcal{A}$ *lies on the imaginary axis. If* $\sigma(\mathcal{A}) \cap i\mathbb{R}$ *is countable, then* T *is stable.*

Indeed, in our case $\mathcal{A}$ has a compact resolvent, which implies that $\sigma(\mathcal{A})$ is purely formed of eigenvalues and the conditions of Theorem 1.3.1 reduce to $\sigma_d(\mathcal{A}) \cap i\mathbb{R} = \phi$.

In order to prove the asymptotic stability of the energy of system (1.1) under some appropriate assumptions, we characterize in the subsequent section the eigenvalues of $\mathcal{A}$.

1.3.1 Characteristic equation

Proposition 1.3.2 *$\mathcal{A}$ is invertible if and only if B is invertible.*

Proof. Let $\begin{pmatrix} y \\ z \\ \eta \end{pmatrix} \in D(\mathcal{A})$ be a solution of

$$\mathcal{A}\begin{pmatrix} y \\ z \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

That is,

$$\begin{pmatrix} z \\ y_{xx} \\ B\eta + Cz(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

which is equivalent to

$$y_{xx} = 0, z = 0, y(0) = 0, B\eta = 0, y_x(1) + (\eta, C) = 0.$$

We write equivalently

$$y = cx, z = 0, B\eta = 0, c + (\eta, C) = 0,$$

for some $c \in \mathbb{C}$.
Suppose $0 \notin \sigma(B)$. Then $\eta = 0$, which implies $c = 0$, hence $0 \notin \sigma(\mathcal{A})$.
Suppose $0 \in \sigma(B)$. Choose η an eigenvector of B, then $(-(\eta, C)x, 0, \eta)^T$ is an eigenvector of $\mathcal{A}$ associated with 0. Thus $0 \in \sigma(\mathcal{A})$. ∎

Proposition 1.3.3 *A complex number λ is an eigenvalue of $\mathcal{A}$ if and only if it satisfies the characteristic equation given by:*

$$C_{\mathcal{A}}(\lambda) = \det\begin{pmatrix} \lambda I - B & -C\sinh\lambda \\ C^* M & \cosh\lambda \end{pmatrix} = 0.$$

Proof. Let λ be a non zero eigenvalue of $\mathcal{A}$. Let $\begin{pmatrix} y \\ z \\ \eta \end{pmatrix} \in D(\mathcal{A})$ be the associated eigenvector.
Then, we have

$$\begin{pmatrix} y \\ z \\ \eta \end{pmatrix} \in D(\mathcal{A}), \quad \mathcal{A}\begin{pmatrix} y \\ z \\ \eta \end{pmatrix} = \lambda\begin{pmatrix} y \\ z \\ \eta \end{pmatrix},$$

equivalently,

$$\begin{pmatrix} y \\ z \\ \eta \end{pmatrix} \in D(\mathcal{A}), \quad \begin{pmatrix} z \\ y_{xx} \\ B\eta + Cz(1) \end{pmatrix} = \lambda\begin{pmatrix} y \\ z \\ \eta \end{pmatrix}.$$

Thus $z = \lambda y, y_{xx} = \lambda z, B\eta + Cz(1) = \lambda\eta, y(0) = 0, z(0) = 0, y_x(1) + (\eta, C) = 0$.
We get

$$y_{xx} = \lambda^2 y, y(0) = 0, z = \lambda y, (\lambda I - B)\eta - C\lambda y(1) = 0, C^* M\eta + y_x(1) = 0. \tag{1.7}$$

Then there exists $\alpha \in \mathbb{C}$ such that

$$y(x) = \alpha \sinh(\lambda x), z(x) = \alpha\lambda \sinh(\lambda x), (\lambda I - B)\eta - C\lambda \sinh(\lambda)\alpha = 0, C^* M\eta + \lambda \cosh(\lambda)\alpha = 0. \tag{1.8}$$

Hence to find $\lambda \in \sigma(\mathcal{A})$ is equivalent to find a nonzero couple (η, α) solution of

$$\begin{pmatrix} \lambda I - B & -C\lambda \sinh \lambda \\ C^* M & \lambda \cosh \lambda \end{pmatrix} \begin{pmatrix} \eta \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Consequently, $\lambda \in \sigma(\mathcal{A})$ if and only if $C_{\mathcal{A}}(\lambda) = \det \begin{pmatrix} \lambda I - B & -C \sinh \lambda \\ C^* M & \cosh \lambda \end{pmatrix} = 0.$

Remark that $C_{\mathcal{A}}(0) = \det \begin{pmatrix} -B & 0 \\ C^* M & 1 \end{pmatrix} = \det(-B)$. Thus 0 is a root of $C_{\mathcal{A}}$ if and only if 0 is an eigenvalue of B, i.e. an eigenvalue of $\mathcal{A}$ by Proposition 1.3.2 . ■

Proposition 1.3.4 *Let $\lambda \notin \sigma(B)$ be nonzero. Then $\lambda \in \sigma(A)$ if and only if λ satisfies*

$$\cosh \lambda + \left((\lambda I - B)^{-1} C, C\right)_{\mathbb{C}^n} \sinh \lambda = 0, \tag{1.9}$$

or equivalently

$$\det(\lambda I - B) \cosh \lambda + (adj(\lambda I - B)C, C)_{\mathbb{C}^n} \sinh \lambda = 0, \tag{1.10}$$

where $adj(\lambda I - B)$ denotes the adjugate matrix of $\lambda I - B$.

Proof. Let $(y, z, \eta)^\top \in D(\mathcal{A})$ be the associated eigenvector. Using equation (1.7) there exists $\alpha \in \mathbb{C}$ such that y, z and η satisfy (1.8).
Since $(\lambda I - B)$ is invertible, then supposing $\alpha = 0$ implies that $(y, z, \eta) = (0, 0, 0)$. We deduce that $\alpha \neq 0$ and as $\lambda \notin \sigma(B)$ we can write

$$\eta = (\lambda I - B)^{-1} C\alpha\lambda \sinh \lambda,$$

thus λ satisfies

$$\lambda \cosh \lambda + ((\lambda I - B)^{-1} C, C)_{\mathbb{C}^n} \lambda \sinh \lambda = 0.$$

Hence every nonzero $\lambda \notin \sigma(B)$ satisfies the characteristic equation given by (1.9). Noting that $(\lambda I - B)^{-1} = \frac{1}{\det(\lambda I - B)} \mathrm{adj}(\lambda I - B)$, we may write our characteristic equation satisfied by any nonzero $\lambda \in \sigma(\mathcal{A}) \backslash \sigma(B)$ as in equation (1.10). ■

Up to now we obtained the characteristic equation satisfied by all $\lambda \in \sigma(\mathcal{A})$, and gave a precise characterization of $\lambda \in \sigma(\mathcal{A})$ in case it is not an eigenvalue of B. The next section is dedicated to the discussion of the conditions that allows to obtain asymptotic stability of the C_0-semigroup generated by $\mathcal{A}$.

1.3.2 Conditions of stability

Proposition 1.3.5 *Let $z \in \mathbb{R}^*$. Then $iz \notin \sigma(B)$ is an eigenvalue of $\mathcal{A}$ if and only if*

$$\Re((izI - B)^{-1} C, C)_{\mathbb{C}^n} = 0 \text{ and } \cos z - \Im((izI - B)^{-1} C, C)_{\mathbb{C}^n} \sin z = 0. \tag{1.11}$$

Proof. Indeed, by (1.9), we have

$$iz \in \sigma(\mathcal{A}) \setminus \sigma(B) \Leftrightarrow \cos z + i \sin z \left(\Re\left((izI - B)^{-1}C, C\right)_{\mathbb{C}^n} + i\Im\left((izI - B)^{-1}C, C\right)_{\mathbb{C}^n}\right) = 0,$$

which is equivalent to

$$\begin{cases} \Re((izI - B)^{-1}C, C)_{\mathbb{C}^n} \sin z & = & 0, \\ \cos z - \Im\left((izI - B)^{-1}C, C\right)_{\mathbb{C}^n} \sin z & = & 0. \end{cases} \tag{1.12}$$

If $\sin z = 0$, then $\cos z \neq 0$ and system (1.12) has no solution. We deduce that z is a solution of (1.12) if and only if (1.11) holds. ∎

Proposition 1.3.6 *For any integer k, $ik\pi$ is an eigenvalue of $\mathcal{A}$ if and only if $ik\pi$ is an eigenvalue of B.*

Proof. Being an eigenvalue of $\mathcal{A}$, $ik\pi$ satisfies $C_{\mathcal{A}}(ik\pi) = 0$, i.e.

$$\pm \det(ik\pi I - B) = \det \begin{pmatrix} ik\pi I - B & 0 \\ C^* M & (-1)^k \end{pmatrix} = 0,$$

that is $ik\pi \in \sigma(B)$. ∎

Proposition 1.3.7 *Let $\lambda \in \sigma(B)$ such that $\lambda \neq ik\pi$ for every $k \in \mathbb{Z}$. Suppose that all nonzero $\eta \in \ker(\lambda I - B)$ are not orthogonal to C. Then $C \notin \ker(\bar{\lambda} I - B^*)^\perp$ implies that $\lambda \notin \sigma(\mathcal{A})$.*

Proof. Indeed, $\lambda \in \sigma(\mathcal{A})$ implies the existence of $(\alpha, \eta) \neq (0, 0)$ satisfying

$$(\lambda I - B)\eta = C\lambda \sinh \lambda\alpha \text{ and } \alpha\lambda \cosh \lambda + (\eta, C) = 0.$$

If $\alpha = 0$ then $0 \neq \eta \in \ker(\lambda I - B)$ and $(\eta, C) = 0$. Hence C is orthogonal to η.
If $\alpha \neq 0$ then $C \in \mathrm{Im}(\lambda I - B) = \ker(\bar{\lambda} I - B^*)^\perp$. ∎

Proposition 1.3.8 *The following assumptions are sufficient to obtain stability of the C_0-semigroup associated with $\mathcal{A}$*

(A_1) $\Re((izI - B)^{-1}C, C) > 0, \forall iz \notin \sigma(B), z \in \mathbb{R}^*$.
(A_2) $ik\pi \notin \sigma(B), \forall k \in \mathbb{Z}$.
(A_3) $\forall iz \in \sigma(B), C \notin \ker(izI + B^*)^\perp$ and $(\eta, C) \neq 0$ for all nonzero $\eta \in \ker(izI - B)$.

Proof. The proof follows directly from Proposition 1.3.5, Proposition 1.3.6, Proposition 1.3.7 and the theorem of Arendt-Batty. ∎

Definition 1.3.9 *A matrix B is said to be Hurwitz if all its eigenvalues have negative real parts.*

Corollary 1.3.10 *Assume B is Hurwitz and Condition (A_1) holds, then the C_0-semigroup generated by $\mathcal{A}$ is strongly stable.*

Decompose B into a sum of a skew-adjoint matrix $B_0 = \frac{B-B^*}{2}$ and a self-adjoint matrix $R = \frac{B+B^*}{2}$, where B^* is the adjoint matrix of B with respect to the inner product $(\cdot, \cdot)_{\mathbb{C}^n}$.

1.4 Polynomial stability

Let us first define the conservative system associated with system (1.1) by

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) & = & 0, & 0 < x < 1, t > 0, \\ y(0,t) & = & 0, & t > 0, \\ y_x(1,t) + (\eta(t), C)_{\mathbb{C}^n} & = & 0, & t > 0, \\ \eta_t(t) - B_0\eta(t) - Cy_t(1,t) & = & 0, & t > 0, \end{cases} \tag{1.13}$$

with the initial conditions given by (1.2). Let A be defined by $D(A) = D(\mathcal{A})$ and

$$A \begin{pmatrix} y \\ z \\ \eta \end{pmatrix} = \begin{pmatrix} z \\ y_{xx} \\ B_0\eta + Cz(1) \end{pmatrix}, \forall \begin{pmatrix} y \\ z \\ \eta \end{pmatrix} \in D(\mathcal{A}).$$

1.4.1 Non uniform stability of $\mathcal{A}$

We present in the subsequent lemma the tool to be used in proving the non uniformity of the C_0-semigroup generated by $\mathcal{A}$ (See for instance [44]).

Lemma 1.4.1 *Let $A = -A^*$ be the infinitesimal generator of a C_0 group, and let B be a compact operator in the Hilbert space H. Then the group $(T(t))_{t>0}$, generated by the operator $-(A+B)$, has no uniform energy decay rate for $t > 0$.*

By replacing B by B_0 in the proof of Proposition 1.2.1 we deduce that for $\lambda > 0$, $\lambda I - A$ is surjective. Also, using the same method as in the proof of Proposition 1.2.1 we can easily show that $\lambda I + A$ is surjective. Accordingly, the subsequent remark follows.

Remark 1.4.2 *Since B_0 is skew-adjoint, it follows that A is skew-symmetric. Now, as $\lambda I - A$ and $\mu I + A$ are onto for some $\lambda > 0$, $\mu > 0$, A is skew-adjoint. According to Stone's theorem (see Theorem* 10.8 *in Chapter* 1 *of [42]), A generates a unitary group.*

Proposition 1.4.3 *The C_0-semigroup associated with $\mathcal{A}$ is not uniformly stable.*

Proof. We have $(\mathcal{A} - A)\begin{pmatrix} y \\ z \\ \eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ R\eta \end{pmatrix}$, which is compact since it is a finite rank operator. The required result follows from Lemma 1.4.1. ■

Next, we discuss the asymptotic behavior of the eigenvalues and eigenvectors of A. Later, we establish some inequalities corresponding to the solutions of system (1.1) and those of the conservative one (1.13) to deduce finally the polynomial stability.

1.4.2 Asymptotic behavior of the spectrum of A

Denote by $P : \mathbb{C}^n \to W$ the projection map from $\mathbb{C}^n$ onto W, with $W = (\ker R)^{\perp}$. We recall that P is linear continuous map satisfying $P^2 = P$ and that any $\eta \in \mathbb{C}^n = \ker R \oplus W$ can be written in a unique way as $\eta = P\eta + \tilde{\eta}$, where $P\eta \in W$ and $\tilde{\eta} \in \ker R$.

Remark 1.4.4 *Clearly, $(-R\eta,\eta)^{\frac{1}{2}}$ is a norm on W. As W is a finite dimensional space, it follows from the property of the equivalence of norms on finite dimensional spaces that there exists $\alpha>0$ such that $(-R\eta,\eta)\geq\alpha\|P\eta\|_2^2$, for all $\eta\in\mathbb{C}^n$, where $\|\cdot\|_2$ is the Euclidean norm.*

As $\Re(Au,u)=0,$ for all $u\in D(A)$, then all the eigenvalues of A are purely imaginary. Denote by $\lambda=i\mu$ the eigenvalues of A, by $\phi_\mu=(y_\mu,z_\mu,\eta_\mu)$ the associated eigenvectors. Due to Proposition 1.3.4, an associated eigenvector with $\lambda=i\mu$, where $|\mu|>\|B_0\|$, is given by

$$\phi_\mu=\frac{1}{\sqrt{N(\mu)}}(i\sin(\mu x),-\mu\sin(\mu x),-(i\mu I-B_0)^{-1}C\mu\sin\mu),$$

with $N(\mu)$ the factor of normalization given by

$$\begin{aligned}N(\mu)&=\mu^2+\|(i\mu I-B_0)^{-1}C\|^2_{\mathbb{C}^n}\mu^2\sin^2(\mu)\\&=\mu^2+\sin^2(\mu)\|(I-\frac{B_0}{i\mu})^{-1}C\|^2_{\mathbb{C}^n}.\end{aligned}$$

Assume that there exists $p\in\mathbb{N}\cup\{0\}$ such that $P(B_0^pC)\neq 0$. Let

$$m=\min\{p\in\mathbb{N}\cup\{0\}:P(B_0^pC)\neq 0\}. \tag{1.14}$$

Proposition 1.4.5 *As $k\to\infty$, the asymptotic expansions of μ_k (with $\Im(\mu_k)\in(k\pi,(k+1)\pi)$), $N(\mu_k)$ and $P\eta_{\mu_k}$ are given by*

$$\mu_k=k\pi+\frac{\pi}{2}+\frac{\|C\|^2}{k\pi}-\frac{\|C\|^2}{2k^2\pi}+\frac{(B_0C,C)}{ik^2\pi^2}+o\left(\frac{1}{k^2}\right),$$

$$N(\mu_k)=k^2\pi^2+o(k^2),$$

$$P\eta_{\mu_k}=(-1)^k\frac{1}{i^{m-1}}\left(\frac{P(B_0^mC)}{k^{m+1}\pi^{m+1}}+o\left(\frac{1}{k^{m+1}}\right)\right).$$

Moreover, the expansions of μ_{-k} (with $\Im(\mu_{-k})\in(-(k+1)\pi,-k\pi)$) and $P\eta_{\mu_{-k}}$ are

$$\mu_{-k}=-k\pi-\frac{\pi}{2}-\frac{\|C\|^2}{k\pi}+\frac{\|C\|^2}{2k^2\pi}+\frac{(B_0C,C)}{ik^2\pi^2}+o\left(\frac{1}{k^2}\right),$$

$$P\eta_{\mu_{-k}}=(-1)^{k-m+1}\frac{1}{i^{m-1}}\left(\frac{P(B_0^mC)}{k^{m+1}\pi^{m+1}}+o\left(\frac{1}{k^{m+1}}\right)\right).$$

Proof. For $\mu\in\mathbb{C}$ with $|\mu|$ large enough, namely $|\mu|>\|B_0\|$ and using Proposition 1.3.4 we have that $i\mu$ is an eigenvalue of $\mathcal{A}$ if and only if μ satisfies

$$\cos\mu+\left((i\mu I-B_0)^{-1}C,C\right)_{\mathbb{C}^n}i\sin\mu=0,$$

which implies

$$\cos\mu=-i\left((i\mu I-B_0)^{-1}C,C\right)_{\mathbb{C}^n}\sin\mu.$$

For $\mu=k\pi$ the above expression is not satisfied, so $\mu\neq k\pi$, dividing by $\sin\mu$, we obtain:

$$\cot\mu=g(\mu),$$

with

$$g(\mu) = -\frac{1}{\mu}\left(\left(I - \frac{B_0}{i\mu}\right)^{-1} C, C\right)_{\mathbb{C}^n}.$$

Since $|\mu| > \|B_0\|$, $I - \frac{B_0}{i\mu}$ is invertible and $\left(I - \frac{B_0}{i\mu}\right)^{-1} = \sum_{n=0}^{\infty}\left(\frac{B_0}{i\mu}\right)^n$.
By the continuity of the mapping $u \to (u, C)(u \in \mathbb{C}^n)$, we get

$$g(\mu) = -\frac{1}{\mu}\left(\sum_{n=0}^{\infty}\left(\frac{B_0}{i\mu}\right)^n C, C\right)_{\mathbb{C}^n} = \sum_{n=0}^{\infty}\left(\frac{-B_0{}^n}{i^n\mu^{n+1}}C, C\right)_{\mathbb{C}^n}.$$

We proceed by studying the variation of g. Indeed, the derivative of g is given by

$$g'(\mu) = \sum_{n=0}^{\infty}\frac{n+1}{\mu^{n+2}}\frac{(B_0^n C, C)}{i^n}.$$

Now, let us discuss the number of roots of g between $k\pi$ and $(k+1)\pi$. For $|\mu|$ large enough, we have

$$g(\mu) = \frac{-\|C\|^2}{\mu} + o\left(\frac{1}{\mu}\right), \text{ and } g'(\mu) = \frac{\|C\|^2}{\mu^2} + o\left(\frac{1}{\mu^2}\right)$$

which implies that g is negative and increasing for such values of μ, thus we deduce that for k large enough there exists a unique μ_k between $k\pi$ and $(k+1)\pi$ solution of $g(\mu) = \cot(\mu)$.
Hence the form of an eigenvalue between $ik\pi$ and $i(k+1)\pi$, is $i\mu_k$ with $\mu_k = k\pi + \frac{\pi}{2} + \varepsilon$. Since $\cot(\mu_k) = g(\mu_k)$, then $\frac{\pi}{2} + \varepsilon = \cot^{-1}(g(\mu_k))$, but $g(\mu) \to 0$ as $\mu \to \infty$, so $\varepsilon = o(1)$. To find the asymptotic expansion of ε, we consider the inverse cotangent of $g(\mu)$. Indeed, we have:

$$\begin{aligned} \cot^{-1}(g(\mu_k)) &= \frac{\pi}{2} - \sum_{k=0}^{\infty}\frac{(-1)^k(g(\mu_k))^{2k+1}}{2k+1} \\ &= \frac{\pi}{2} - g(\mu_k) + o\left((g(\mu_k))^2\right). \end{aligned}$$

But,

$$g(\mu_k) = -\frac{\|C\|^2}{\mu_k} - \frac{(B_0C, C)}{i\mu_k^2} + o\left(\frac{1}{\mu_k^2}\right),$$

$$\frac{1}{\mu_k} = \frac{1}{k\pi} - \frac{1}{2k^2\pi} + o\left(\frac{1}{k^2}\right),$$

$$\frac{1}{\mu_k^2} = \frac{1}{k^2\pi^2} + o\left(\frac{1}{k^2}\right),$$

thus

$$g(\mu_k) = -\frac{\|C\|^2}{k\pi} + \frac{\|C\|^2}{2k^2\pi} - \frac{(B_0C, C)}{ik^2\pi^2} + o\left(\frac{1}{k^2}\right).$$

We finally get the asymptotic expansion of $\mu_k \in (k\pi, (k+1)\pi)$,

$$\mu_k = k\pi + \frac{\pi}{2} + \frac{\|C\|^2}{k\pi} - \frac{\|C\|^2}{2k^2\pi} + \frac{(B_0C, C)}{ik^2\pi^2} + o\left(\frac{1}{k^2}\right).$$

As $\sin(\mu_k) = \sin(k\pi + \frac{\pi}{2} + o(1)) = (-1)^k + o(1)$ and $\|C + \sum_{n=1}^{\infty} \frac{{B_0}^n}{i^n \mu_k^n} C\|^2$ is bounded, we get

$$N(\mu_k) = k^2\pi^2 + o(k^2).$$

The expansion of η_{μ_k} is given by

$$\begin{aligned} \eta_{\mu_k} &= i\frac{\sin \mu_k}{\sqrt{N(\mu_k)}} \left(I - \frac{B_0}{i\mu_k}\right)^{-1} C \\ &= (-1)^k i \left(\frac{1}{k\pi} + o\left(\frac{1}{k}\right)\right) \sum_{n=0}^{\infty} \frac{B_0^n C}{i^n \mu_k^n}. \end{aligned}$$

Taking the projection P of η_{μ_k} on W, we get

$$P\eta_{\mu_k} = (-1)^k \frac{1}{i^{m-1}} \left(\frac{P(B_0^m C)}{k^{m+1}\pi^{m+1}} + o\left(\frac{1}{k^{m+1}}\right)\right).$$

We proceed by the same way to prove the existence of a unique $\mu_{-k} \in [-(k+1)\pi, -k\pi]$ and obtain its asymptotic expansion. ■

1.4.3 An a priori estimate

Let u_1 be the solution of the conservative problem, i.e. u_1 satisfies

$$\left\{ \begin{array}{rcl} \frac{d}{dt}u_1(t) &=& Au_1(t), \quad t > 0, \\ u_1(0) &=& u_0. \end{array} \right. \tag{1.15}$$

Let u be a solution of the original system associated with the coupled PDE-ODE system

$$\left\{ \begin{array}{rcl} \frac{d}{dt}u(t) &=& \mathcal{A}u(t), \quad t > 0, \\ u(0) &=& u_0, \end{array} \right. \tag{1.16}$$

where $D(A) = D(\mathcal{A})$ was introduced before.

Setting $u_2 = u - u_1$, u_2 fulfills the following non homogeneous initial value problem

$$\left\{ \begin{array}{rcl} \frac{d}{dt}u_2(t) &=& Au_2(t) + f(t), \quad t > 0, \\ u_2(0) &=& 0, \end{array} \right. \tag{1.17}$$

where $f = (0, 0, R\eta)$ with η the last component of u.

Definition 1.4.6 *The function $u_2 \in C([0,T]; \mathcal{H})$ given by: $u_2(t) = \int_0^t e^{A(t-s)} f(s)ds$ is the mild solution of the initial value problem (1.17) on $[0,T]$.*

Back to our problem, we have $\frac{d}{dt}E(t) = \Re(B\eta, \eta) = (R\eta, \eta) \leq 0$, then integrating between 0 and any $T > 0$, we get

$$E(T) - E(0) = \int_0^T (R\eta, \eta)dt.$$

We claim that there exists $c > 0$ depending on T such that $c\int_0^T (-R\eta, \eta)dt \geq \int_0^T (-R\eta_1, \eta_1)dt$. To get the polynomial stability, we will impose assumptions to bound $\int_0^T (-R\eta_1, \eta_1)dt$ from below.

Proposition 1.4.7 *For all $T > 0$, there exists $c > 0$ depending on T such that*

$$\int_0^T (-R\eta_1(t), \eta_1(t))dt \leq c \int_0^T (-R\eta(t), \eta(t))dt. \tag{1.18}$$

Proof. Throughout the proof, we use the notation $A \lesssim B$ to denote the inequality $A \leq cB$ for some positive constant c depending on T. Recall that B can be written as $B_0 + R$ where B_0 is skew-adjoint and R is self-adjoint with

$$(R\eta, \eta) = \Re(B\eta, \eta) \leq 0.$$

Note also that $\Re(R\eta, \kappa) = \Re(R\kappa, \eta)$ from which we deduce that,

$$\Re(R\eta, \kappa) \leq (-R\eta, \eta)^{\frac{1}{2}}(-R\kappa, \kappa)^{\frac{1}{2}}.$$

Hence $(-R\eta, \eta)^{\frac{1}{2}}$ defines a semi norm on $\mathbb{C}^n$. As $\eta_1 = \eta - \eta_2$, we get:

$$(-R\eta_1, \eta_1) \leq 2\left((-R\eta, \eta) + (-R\eta_2, \eta_2)\right). \tag{1.19}$$

Next, we show that $\int_0^T (-R\eta_2(t), \eta_2(t))\, dt \lesssim \int_0^T (-R\eta(t), \eta(t))dt$. Indeed, we have:

$$u_2(t) = \int_0^t e^{A(t-s)} \begin{pmatrix} 0 \\ 0 \\ R\eta(s) \end{pmatrix} ds.$$

As A is skew-adjoint, it follows from Remark 1.4.2 that A generates a group. Moreover, we have

$$\begin{aligned}
(u_2(t), u_2(t))_{\mathcal{H}} &= \int_0^t \left(e^{A(t-s)} \begin{pmatrix} 0 \\ 0 \\ R\eta(s) \end{pmatrix}, \begin{pmatrix} y_2(t) \\ z_2(t) \\ \eta_2(t) \end{pmatrix} \right)_{\mathcal{H}} ds \\
&= \int_0^t \left(\begin{pmatrix} 0 \\ 0 \\ R\eta(s) \end{pmatrix}, e^{-A(t-s)} \begin{pmatrix} y_2(t) \\ z_2(t) \\ \eta_2(t) \end{pmatrix} \right)_{\mathcal{H}} ds \\
&= \int_0^t \left(R\eta(s), p_3(e^{-A(t-s)}u_2(t)) \right)_{\mathbb{C}^n} ds,
\end{aligned}$$

where $p_3(u)$ denotes the projection of $u \in \mathcal{H}$ on $\mathbb{C}^n$. It follows that

$$\begin{aligned}
\int_0^t (R\eta(s), p_3(e^{-A(t-s)}u_2(t)))ds &\leq \int_0^t (-R\eta(s), \eta(s))^{\frac{1}{2}}(-Rp_3(e^{-A(t-s)}u_2(t)), p_3(e^{-A(t-s)}u_2(t)))^{\frac{1}{2}} ds \\
&\lesssim \left(\int_0^t (-R\eta(s), \eta(s))ds \right)^{\frac{1}{2}} \left(\int_0^t \|p_3(e^{-A(t-s)}u_2(t))\|^2_{\mathbb{C}^n} ds \right)^{\frac{1}{2}}.
\end{aligned}$$

We then get for $t < T$,

$$\|u_2(t)\|^2_{\mathcal{H}} \lesssim \left(\int_0^t (-R\eta(s), \eta(s))\, ds \right)^{\frac{1}{2}} \|u_2(t)\|_{\mathcal{H}}$$

Consequently,

$$\|\eta_2(t)\|_{\mathbb{C}^n} \leq \|u_2(t)\|_{\mathcal{H}} \lesssim \left(\int_0^T (-R\eta(t), \eta(t))\, dt\right)^{\frac{1}{2}},$$

which leads to

$$\int_0^T (-R\eta_2(t), \eta_2(t))\, dt \lesssim \int_0^T \|\eta_2(t)\|^2_{\mathbb{C}^n} dt \lesssim \int_0^T (-R\eta(t), \eta(t))\, dt.$$

Hence by (1.19), we conclude that (1.18) holds. ■

In the next section we find a lower bound of $\int_0^T (-R\eta_1(t), \eta_1(t))\, dt$, that allows to deduce a lower bound of $\int_0^T (-R\eta(t), \eta(t))\, dt$.

1.4.4 An Observability Inequality

First, we present the following lemma which guarantees that $P\eta_\mu \neq 0$, for all μ with small modulus. The purpose of this lemma is to deal with the terms of low indices in the proposition that follows. As a second consequence, it also implies that under its assumptions $\mathcal{A}$ has no eigenvalues on the imaginary axis.

Lemma 1.4.8 *Suppose that $\sigma(A) \cap \sigma(B_0) \subset \{ik\pi : k \in \mathbb{Z}\}$ and $P\nu_k \neq 0$, where ν_k represents the eigenvector of B_0 associated with $ik\pi \in \sigma(B_0)$. Suppose moreover that*

$$P\left((i\mu I - B_0)^{-1}C\right) \neq 0, \forall i\mu \in \sigma(A)\backslash\sigma(B_0),$$

then $P\eta_\mu \neq 0$, for all $i\mu \in \sigma(A)$.

Proof. Let $i\mu \in \sigma(A)$. If $i\mu \notin \sigma(B_0)$, then by Proposition 1.3.2 and Proposition 1.3.6, we get $i\mu \neq ik\pi$ for all $k \in \mathbb{Z}$. Since η_μ is given by $\eta_\mu = -(i\mu I - B_0)^{-1}C\alpha\mu \sin\mu$ for some nonzero α and $P\left((i\mu I - B_0)^{-1}C\right) \neq 0$, then $P(\eta_\mu) \neq 0$.
If $i\mu \in \sigma(B_0)$, then $\mu = k_0\pi$ for some $k_0 \in \mathbb{Z}$. But by Proposition 1.3.2 and by (1.8) of Proposition 1.3.3, η_μ the last component of the eigenvector ϕ_μ of A associated with $i\mu$ is different from 0 and satisfies $B_0\eta_\mu = i\mu\eta_\mu$. Hence $P\eta_\mu \neq 0$. ■

Remark 1.4.9 *If $C \notin \ker(i\mu I - B_0)^\perp$ and $(\eta, C) \neq 0$ for all nonzero $\eta \in \ker(i\mu I - B_0)$, for all $i\mu \in \sigma(B_0)$ with $\mu \neq k\pi$ for every $k \in \mathbb{Z}$, then by Proposition 1.3.7 we get $\sigma(A) \cap \sigma(B_0) \subset \{ik\pi : k \in \mathbb{Z}\}$. Remark also that if the eigenvalues of B_0 are geometrically simple then the conditions needed to imply $\sigma(A) \cap \sigma(B_0) \subset \{ik\pi : k \in \mathbb{Z}\}$ reduce to $C \notin \ker(i\mu I - B_0)^\perp$ for all $i\mu \in \sigma(B_0)$.*

Corollary 1.4.10 *Under the assumptions of Lemma 1.4.8, we have*

$$\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset.$$

Proof. By Proposition 1.3.3 (see (1.8)), $i\mu \neq 0 \in \sigma(\mathcal{A}) \cap i\mathbb{R}$ if and only if there exists a non zero $(\eta, \alpha) \in \mathbb{C}^n \times \mathbb{C}$ solution of

$$(i\mu I - B_0 - R)\eta + C\mu \sin\mu\alpha = 0, (\eta, C) + i\mu \cos\mu\alpha = 0. \tag{1.20}$$

Taking the inner product of the first identity with η, we find

$$((i\mu I - B_0 - R)\eta, \eta) + \mu \sin \mu \alpha (C, \eta) = 0.$$

Taking into account the second identity of (1.20), we get

$$((i\mu I - B_0 - R)\eta, \eta) + i\mu^2 \sin \mu \cos \mu |\alpha|^2 = 0.$$

The real part of this identity is nothing but

$$-(R\eta, \eta) = 0.$$

Moreover, if $0 \in \sigma(\mathcal{A}) \cap i\mathbb{R}$ then by Proposition 1.3.2 we deduce that $B\eta = 0$ from which we also deduce that $-(R\eta, \eta) = 0$, or equivalently $\eta \in \ker R$ (or $P\eta = 0$). Coming back to (1.20), we see that $(\eta, \alpha) \in \mathbb{C}^n \times \mathbb{C}$ is also solution of

$$(i\mu I - B_0)\eta + C\mu \sin \mu\alpha = 0, (\eta, C) + i\mu \cos \mu\alpha = 0,$$

for $\mu \neq 0$ and $B_0\eta = 0$ for $\mu = 0$. In other words, $i\mu \in \sigma(A)$ and by Lemma 1.4.8, $P\eta$ cannot be zero. ■

We recall in the subsequent theorem an inequality of Ingham's type (see for instance [11]).

Theorem 1.4.11 *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a strictly increasing sequence of real numbers and let U be a separable Hilbert space. Suppose the sequence (λ_n) satisfies the "gap" condition*

$$\exists \gamma > 0, \forall n \in \mathbb{Z}, \lambda_{n+1} - \lambda_n \geq \gamma,$$

then for all sequence $(a_n)_{n\in\mathbb{Z}} \subset U$, the function

$$f(t) = \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t}$$

satisfies the estimate

$$\int_0^T \|f(t)\|^2 \sim \sum_{n\in\mathbb{Z}} \|a_n\|_U^2,$$

for $T > 2\pi\gamma$.

Now we suppose that

$$\text{all } \lambda \in \sigma(A) \cap \sigma(B_0) \subset \{in\pi : n \in \mathbb{Z}\} \text{ are simple eigenvalues of } B_0. \tag{1.21}$$

We already know that the algebraic and the geometric multiplicities of the eigenvalues of A are equal, since A is skew-adjoint. Moreover, we previously showed that the eigenvalues of A which do not belong to $\sigma(B_0)$ are simple. Hence we notice that the algebraic multiplicity of any eigenvalue of A and its geometric multiplicity is equal to 1. As A is a skew-adjoint operator with compact resolvent, the spectrum of A may be represented by a sequence $(\lambda_{0,n})_{n\in I} = (i\mu_n)_{n\in I}$ with $(\mu_n)_{n\in I}$ a strictly increasing sequence, where $I = \mathbb{Z}^*$ if A is invertible and $I = \mathbb{Z}$ with $\mu_0 = 0$ if A is not invertible. Denote by $(\phi_{0,n})_{n\in I} = \left((y_1^{(n)}, z_1^{(n)}, \eta_1^{(n)})^\top\right)_{n\in I}$ the corresponding sequence of eigenvectors associated with $(\lambda_{0,n})_{n\in I}$.

Proposition 1.4.12 *Assume (1.14) and (1.21) and let $u_1 = (y_1, z_1, \eta_1)^T$ be the solution of the conservative problem (1.15) with initial datum $u_0 \in D(\mathcal{A})$. If the assumptions of Lemma 1.4.8 hold, then there exists $T > 0$ and $c > 0$ depending on T such that*

$$\int_0^T \|P\eta_1(t)\|^2 dt \geq c\|u_0\|^2_{D(A^{-(m+1)})}. \tag{1.22}$$

Proof. As $(\phi_{0,n})_{n\in I}$ forms a Hilbert basis of $\mathcal{H}$, we may write $u_0 = \sum_{n\in I} u_0^{(n)} \phi_{0,n}$. Moreover,

$$u_1(t) = \sum_{n\in I} u_0^{(n)} e^{i\mu_n t}\phi_{0,n}, \quad \eta_1(t) = \sum_{n\in I} u_0^{(n)} e^{i\mu_n t}\eta_1^{(n)}, \quad P\eta_1(t) = \sum_{n\in I} u_0^{(n)} e^{i\mu_n t} P\eta_1^{(n)}.$$

Note that $\mu_{n+1} - \mu_n \geq \frac{\pi}{2}$ for $|n|$ large enough, say for $|n| \geq n_0$. Set $\gamma_0 = \min\left\{\frac{\pi}{2}, \min\{\mu_{n+1} - \mu_n : |n| < n_0\}\right\}$. As $\mu_{k+1} - \mu_k \geq \gamma_0 > 0$, then using Ingham's inequality there exists $T > 2\pi\gamma_0 > 0$ and a constant $c > 0$ depending on T such that

$$\int_0^T \|P\eta_1(t)\|^2 dt \geq c \sum_{n\in I} \|u_0^{(n)} P\eta_1^{(n)}\|^2.$$

For $n_0 \in \mathbb{N}$ large enough, there exists an integer k_{n_0} such that for all $|n| \geq n_0$ we have $\mu_n \in [k_n\pi, k_{n+1}\pi]$ with $k_{n+1} = k_n + 1$ and

$$P\eta_1^{(n)} = (-1)^n \frac{1}{i^{m-1}} \left(\frac{P(B_0^m C)}{k_n^{m+1}\pi^{m+1}} + o\left(\frac{1}{k_n^{m+1}}\right)\right),$$

then using the fact that $\|P\eta_1^{(n)}\|^2 \sim \frac{1}{k_n^{2(m+1)}\pi^{2(m+1)}}$, and due to Lemma 1.4.8, we obtain by Ingham's inequality the existence of $T > 0$ such that

$$\int_0^T \|P\eta_1\|^2 dt \gtrsim \sum_{|n|<n_0} |u_0^{(n)}|^2 |\lambda_{0,n}|^{-2(m+1)} + \sum_{|n|\geq n_0} \frac{|u_0^{(n)}|^2}{k_n^{2(m+1)}}, \tag{1.23}$$

with the notation $0^{-2(m+1)} = 1$. For a non invertible A, define the norm on $D(A^{-(m+1)})$ by

$$\|u_0\|^2_{D(A^{-(m+1)})} := |u_0^{(0)}|^2 + \sum_{n\in Z^*} |\lambda_{0,n}|^{-2(m+1)} |u_0^{(n)}|^2,$$

then estimate (1.23) implies (1.22). ■

1.4.5 Interpolation inequality

Lemma 1.4.13 *For all $u_0 \in D(A)$ and all $s \in \mathbb{N}$, we have $\|u_0\|^{s+1}_{\mathcal{H}} \leq \|u_0\|_{D(A^{-s})} \|u_0\|^s_{D(A)}$.*

Proof. We proceed by proving: $\|u_0\|^2_{\mathcal{H}} \leq \|u_0\|^{\frac{2}{s+1}}_{D(A^{-s})} \|u_0\|^{\frac{2s}{s+1}}_{D(A)}$.
In fact, we have $\|u_0\|^2_{\mathcal{H}} = \sum_{n\in I} |u_0^{(n)}|^2$, where $u_0 = \sum_{n\in I} u_0^{(n)} \phi_{0,n}$ with $I = Z$ or $I = Z^*$.

Using Cauchy-Schwarz inequality, we get for any $\gamma > 0, \beta > 0$ such that $\gamma + \beta = 2$, and any conjugate exponents $p > 0, q > 0$ and any $\alpha > 0$

$$\begin{aligned}\sum_{n\in I} |u_0^{(n)}|^2 &= \sum_{n\in I} |u_0^{(n)}|^2 |\lambda_{0,n}|^{\alpha} |\lambda_{0,n}|^{-\alpha} = \sum_{n\in I} |u_0^{(n)}|^{\beta} |\lambda_{0,n}|^{\alpha} |u_0^{(n)}|^{\gamma} |\lambda_{0,n}|^{-\alpha} \\ &\le \left(\sum_{n\in I} (|u_0^{(n)}|^{\beta} |\lambda_{0,n}|^{\alpha})^p\right)^{\frac{1}{p}} \left(\sum_{n\in I} (|u_0^{(n)}|^{\gamma} |\lambda_{0,n}|^{-\alpha})^q\right)^{\frac{1}{q}},\end{aligned}$$

with $0^{\alpha} = 0^{-\alpha} = 1$ as notation, and $\|u_0\|^2_{D(A^r)} := \sum_{n\in I} |u_0^{(n)}|^2 |\lambda_{0,n}|^{2r}$ for all $r \in \mathbb{R}$.

The result follows from the following choice $q = s + 1$, $p = \frac{q}{s}$, $\alpha = \beta = \frac{2}{p} = \frac{2s}{q}$, $\gamma = \frac{2}{q}$. ■

The next Lemma proved in Lemma 5.2 of [8] is required for the proof of stability.

Lemma 1.4.14 *Let $(\varepsilon_k)_k$ be a sequence of positive real numbers satisfying*

$$\varepsilon_{k+1} \le \varepsilon_k - C\varepsilon_{k+1}^{2+\alpha}, \forall k \ge 0,$$

where $C > 0$ and $\alpha > -1$. Then there exists a positive constant M (depending on α and C) such that

$$\varepsilon_k \le \frac{M}{(1+k)^{\frac{1}{\alpha+1}}}, \forall k \ge 0.$$

1.4.6 The polynomial stability

We are now able to state our polynomial stability result.

Theorem 1.4.15 *Let u be a solution of the problem (1.3) with initial datum $u_0 \in D(\mathcal{A})$. Let the assumptions of Lemma 1.4.8 be satisfied. Assume moreover (1.21) and the existence of m as defined by equation (1.14), then we obtain the following polynomial energy decay:*

$$E(t) \le \frac{M}{(1+t)^{\frac{1}{m+1}}} \|u_0\|^2_{D(\mathcal{A})},$$

for some $M > 0$.

Proof. Let $p = m+1$. With $T > 0$ from Proposition 1.4.12 we have $E(T) \le E(0) - K\|u_0\|^2_{D(A^{-p})}$, where K is a positive constant depending on T. Indeed, by Proposition 1.4.12 we have

$$E(T) - E(0) = \int_0^T (R\eta, \eta)dt \le \frac{1}{c} \int_0^T (R\eta_1, \eta_1)dt \le \frac{-\alpha}{c} \int_0^T \|P\eta_1(t)\|_2^2 dt \le -K\|u_0\|^2_{D(A^{-p})}.$$

Set $E_1(0) = \frac{1}{2}\left(\|u_0\|^2_{\mathcal{H}} + \|Au_0\|^2_{\mathcal{H}}\right)$.

By Lemma 1.4.13, we have $\|u_0\|^2_{D(A^{-p})} \ge \frac{\|u_0\|^{2(p+1)}_{\mathcal{H}}}{\|u_0\|^{2p}_{D(A)}} \ge \frac{E^{p+1}(0)}{E_1^p(0)}$. Since the energy is decreasing with time, we obtain

$$E(T) \le E(0) - K\frac{E^{p+1}(0)}{E_1^p(0)} \le E(0) - K\frac{E^{p+1}(T)}{E_1^p(0)}.$$

We prove similarly that

$$E((k+1)T) \leq E(kT) - K\frac{E^{p+1}((k+1)T)}{E_1^p(kT)}.$$

Dividing by $E_1(0)$ and noting that $E_1(0) \geq E_1(kT)$, we obtain

$$\varepsilon_{k+1} \leq \varepsilon_k - K\varepsilon_{k+1}^{p+1},$$

with $\varepsilon_k = \frac{E(kT)}{E_1(0)}$.
Applying Lemma 1.4.14, there exists $M > 0$ depending on T such that $\varepsilon_k \leq \frac{M}{(1+k)^{\frac{1}{1+\alpha}}}$, where $2+\alpha = p+1$, which implies that

$$E(t) \leq \frac{M}{(1+t)^{\frac{1}{p}}} E_1(0),$$

since $E_1(0) \lesssim \|u_0\|^2_{D(\mathcal{A})}$. ■

1.5 Optimality of the energy decay rate

Let

$$f(\lambda) = \det(\lambda I - B)\cosh\lambda + (\mathrm{adj}(\lambda I - B)C, C)\sinh\lambda,$$
$$g(\lambda) = \det(\lambda I - B_0)\cosh\lambda + (\mathrm{adj}(\lambda I - B_0)C, C)\sinh\lambda.$$

In order to find a correspondence between the eigenvectors of $\mathcal{A}$ and A, we discuss in the subsequent proposition the number of roots of g and f in appropriate regions of the complex plane.

Proposition 1.5.1 *The number of eigenvalues of $\mathcal{A}$ counted with multiplicities is equal to that of A in the square $C_n = [-n\pi, n\pi] \times [-n\pi, n\pi]$, for n large enough.*

Proof. Using Rouché's Theorem we prove that f and g have the same number of roots in C_n, for n large enough.

Let $h(\lambda) = \cosh\lambda + ((\lambda I - B)^{-1}C, C)\sinh\lambda$ and $h_0(\lambda) = \cosh\lambda + ((\lambda I - B_0)^{-1}C, C)\sinh\lambda$.

Computing $h(\lambda) - h_0(\lambda)$ for $|\lambda|$ large enough, we get:

$$\begin{aligned}
h(\lambda) - h_0(\lambda) &= \left((\lambda I - B)^{-1} C, C\right)\sinh\lambda - \left((\lambda I - B_0)^{-1} C, C\right)\sinh\lambda \\
&= \frac{\sinh\lambda}{\lambda}\left(\left(I - \frac{B}{\lambda}\right)^{-1} C - \left(I - \frac{B_0}{\lambda}\right)^{-1} C, C\right)_{\mathbb{C}^n} \\
&= \frac{\sinh\lambda}{\lambda}\left(\sum_{n=1}^{\infty}\frac{B^n - B_0^n}{\lambda^n} C, C\right)_{\mathbb{C}^n} \\
&= \frac{(RC, C)}{\lambda^2}\sinh\lambda + o\left(\frac{1}{\lambda^2}\right)\sinh\lambda.
\end{aligned}$$

Consider the ratio of $|h(\lambda)-h_0(\lambda)|$ by $|h_0(\lambda)|$

$$\begin{aligned}\frac{|h(\lambda)-h_0(\lambda)|}{|h_0(\lambda)|} &= \frac{|\frac{(RC,C)}{\lambda^2}\sinh\lambda + o(\frac{1}{\lambda^2})\sinh\lambda|}{|\cosh\lambda + ((\lambda I-B)^{-1}C,C)\sinh\lambda|}\\ &\leq \frac{K}{|\lambda^2\frac{\cosh\lambda}{\sinh\lambda}+\lambda\sum_{n=0}^{\infty}\frac{(B^nC,C)}{\lambda^n}|}.\end{aligned}$$

Note that for $\lambda = x+iy$, we have $|\sinh\lambda|^2 = \sin^2 y + \sinh^2 x$, and $|\cosh\lambda|^2 = \cos^2 y + \sinh^2 x$, thus $1-\frac{1}{\sinh^2 x} \leq |\frac{\cosh\lambda}{\sinh\lambda}|^2 \leq 1+\frac{1}{\sinh^2 x}$, which implies that for $|x| = |\Re(\lambda)| \to \infty$, $|\frac{\cosh\lambda}{\sinh\lambda}| \to 1$. We deduce that

$$\frac{|h(\lambda)-h_0(\lambda)|}{|h_0(\lambda)|} \to 0, \text{ as } |\Re(\lambda)| \to \infty.$$

Suppose that $|\Im(\lambda)| = |y| = n\pi$, then $|\frac{\lambda\cosh\lambda}{\sinh\lambda}|^2 = (n^2\pi^2+x^2)\left(1+\frac{1}{\sinh^2 x}\right) \geq n^2\pi^2$. It follows that for such λ, $\frac{|h(\lambda)-h_0(\lambda)|}{|h_0(\lambda)|} \to 0$ as $n\to\infty$.
For n chosen large enough, we then have $\frac{|h(\lambda)-h_0(\lambda)|}{|h_0(\lambda)|} \to 0$, for all $\lambda \in \partial C_n$.
Consider the ratio $\frac{|f(z)-g(z)|}{|g(z)|}$, we may write for $|\lambda| > \max\{\|B\|,\|B_0\|\}$

$$\begin{aligned}\frac{|f(\lambda)-g(\lambda)|}{|g(\lambda)|} &= \frac{|\det(\lambda I-B)h(\lambda)-\det(\lambda I-B_0)h_0(\lambda)|}{|\det(\lambda I-B_0)h_0(\lambda)|}\\ &\leq \frac{|\det(\lambda I-B)|}{|\det(\lambda I-B_0)|}\frac{|h(\lambda)-h_0(\lambda)|}{|h_0(\lambda)|} + \frac{|\det(\lambda I-B)-\det(\lambda I-B_0)|}{|\det(\lambda I-B_0)|}\end{aligned}$$

Knowing that $\det(\lambda I-B)$ is a monic polynomial of degree n we get

$$\frac{|\det(\lambda I-B)|}{|\det(\lambda I-B_0)|} \to 1, \text{ as } |\lambda| \to \infty$$

and

$$\frac{|\det(\lambda I-B)-\det(\lambda I-B_0)|}{|\det(\lambda I-B_0)|} \lesssim \frac{1}{|\lambda|} \to 0, \text{ as } |\lambda| \to \infty.$$

We deduce that for $\lambda \in \partial C_n$ with $|\lambda| \to \infty$, we have $\frac{|f(\lambda)-g(\lambda)|}{|g(\lambda)|} \to 0$. Thus for $\lambda \in \partial C_n$ and n large enough, $\frac{|f(\lambda)-g(\lambda)|}{|g(\lambda)|} < 1$.
Clearly, f and g are analytic, this together with the above comparison allows to apply Rouché's theorem in C_n for n large enough, from which we deduce that f and g have the same number of roots in C_n. ■

Definition 1.5.2 *A system $\{g_n\}_{n\in\mathbb{N}}$ of elements of $\mathcal{H}$ is said to be $\omega-$linearly independent if*

$$\sum_{n=1}^{\infty} a_n g_n = 0 \textit{ implies } a_n = 0, \forall n \in \mathbb{N}.$$

In order to show that the generalized eigenfunctions of $\mathcal{A}$ are $\omega-$linearly independent, we use the following Lemma mentioned in [21].

Lemma 1.5.3 *Let B be a densely defined closed linear operator in a Hilbert space $\mathcal{H}$. Assume that the spectrum of B consists entirely of, at most countable, isolated points, each of which has a finite algebraic multiplicity. Then the generalized eigenfunctions are $\omega-$linearly independent.*

To use Lemma 1.5.3, we show that the algebraic multiplicities of all eigenvalues of $\mathcal{A}$ are finite. In fact, the multiple eigenvalues of $\mathcal{A}$ must satisfy a polynomial equation, thus they are finite and of finite algebraic multiplicity.

Proposition 1.5.4 *All the eigenvalues of $\mathcal{A}$ have finite algebraic multiplicities. Moreover, the eigenvalues with large enough moduli are algebraically simple.*

Proof. Define F by:

$$\begin{aligned} F(z) &= 2e^z f(z) \\ &= e^{2z} P_1(z) + P_2(z) \end{aligned}$$

where $P_1(z) = \det(zI - B) + (\mathrm{adj}(zI - B)C, C)$, and $P_2(z) = \det(zI - B) - (\mathrm{adj}(zI - B)C, C)$. Every non simple eigenvalue z satisfies $F(z) = F'(z) = 0$, and thus satisfies $P(z) = 0$, where

$$P(z) = \frac{F'(z)P_2(z) - F(z)P_2'(z)}{e^{2z}} = \left(2P_1(z) + P_1'(z)\right) P_2(z) - P_1(z)P_2'(z).$$

Hence a non simple eigenvalue z of $\mathcal{A}$ is one of at most $2n$ roots of $P(z)$ having a multiplicity $\leq 2n$, where n is the dimension of $\mathbb{C}^n$. ■

Remark 1.5.5 *Assume that*

$$\sigma(A) \cap \sigma(B_0) \subseteq \{ik\pi : k \in \mathbb{Z}\} \text{ and } \sigma(\mathcal{A}) \cap \sigma(B) = \phi, \tag{1.24}$$

then f and g describe the full spectrum of $\mathcal{A}$ and A respectively allowing to get a one-to-one correspondence between the spectra of $\mathcal{A}$ and A. In practice, we will check conditions of Remark 1.4.9, $C \notin \ker(\bar{\lambda} I - B^*)^{\perp}$ and $(\eta, C) \neq 0$ for all nonzero $\eta \in \ker(\lambda I - B)$ for every $\lambda \in \sigma(B)$ to show that g represents the characteristic equation satisfied by all eigenvalues of $\mathcal{A}$. Note that the last condition is equivalent to $C \notin \ker(\lambda I - B)^{\perp}$ for a geometrically simple eigenvalue λ of B.

In what follows we discuss the asymptotic behavior of the eigenvalues of $\mathcal{A}$ and the associated eigenvectors that allows studying some cases in which optimality of the polynomial decay can be obtained.

Proposition 1.5.6 *Let λ be an eigenvalue of $\mathcal{A}$ with $|\lambda|$ large enough. Then λ satisfies the following expansion for some k large enough,*

$$\lambda = i\left(k\pi + \frac{\pi}{2} + \frac{\|C\|^2}{k\pi} - \frac{\|C\|^2}{2k^2\pi} + \frac{(B_0C, C)}{ik^2\pi^2}\right) + \frac{(RC, C)}{k^2\pi^2} + o\left(\frac{1}{k^2}\right). \tag{1.25}$$

Proof. Let λ with $|\lambda| > \|B\|$ be a root of h, since $F(\lambda) = 0$, then

$$e^{2\lambda} = -\left(1 - \frac{2\left((\lambda I - B)^{-1}C, C\right)}{1 + ((\lambda I - B)^{-1}C, C)}\right).$$

Setting

$$x = 2\frac{\big((\lambda I - B)^{-1}C, C\big)}{1 + ((\lambda I - B)^{-1}C, C)},$$

we have

$$\begin{aligned} x &= \frac{2}{\lambda}\sum_{n=0}^{\infty}\frac{(B^n C, C)}{\lambda^n}\frac{1}{1+\displaystyle\sum_{n=0}^{\infty}\frac{(B^n C, C)}{\lambda^{n+1}}} \\ &= 2\frac{\|C\|^2}{\lambda} + 2\frac{(BC, C) - \|C\|^4}{\lambda^2} + o\left(\frac{1}{\lambda^2}\right), \end{aligned}$$

and

$$x^2 = 4\frac{\|C\|^4}{\lambda^2} + o\left(\frac{1}{\lambda^2}\right).$$

It follows that

$$\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n} = -2\frac{\|C\|^2}{\lambda} - 2\frac{(BC, C)}{\lambda^2} + o\left(\frac{1}{\lambda^2}\right).$$

As $|\lambda|$ is large, the imaginary part of λ lies between $k\pi$ and $(k+1)\pi$ for some k large enough, and

$$\lambda = i(k\pi + \frac{\pi}{2}) + \frac{1}{2}\ln(1-x).$$

As $\frac{1}{\lambda} = -\frac{i}{k\pi} + \frac{i}{2k^2\pi} + o\left(\frac{1}{k^2}\right)$ and $\frac{1}{\lambda^2} = -\frac{1}{k^2\pi^2} + o\left(\frac{1}{k^2}\right)$, we obtain (1.25) as required. ■

To show that the system of generalized eigenfunctions of $\mathcal{A}$ forms a Riesz basis of $\mathcal{H}$, we use the following well known Bari's theorem.

Theorem 1.5.7 *Let I be a countable set. Consider the two systems $(\psi_k)_{k\in I}$ and $(\phi_k)_{k\in I}$ of vectors of $\mathcal{H}$ such that $(\phi_k)_{k\in I}$ is a Riesz basis of $\mathcal{H}$. If $(\psi_k)_{k\in I}$ is a sequence of $\omega-$linearly independent vectors quadratically close to $(\phi_k)_{k\in I}$ (i.e. $\sum_{k\in I}\|\psi_k - \phi_k\|^2 < +\infty$), then $(\psi_k)_{k\in I}$ is a Riesz basis of $\mathcal{H}$.*

Denote by $(\lambda_k)_{k\in I}$ the set of eigenvalues of $\mathcal{A}$ counted with multiplicities such that

$$\Im(\lambda_k) \le \Im(\lambda_{k+1}), \forall k \in I.$$

We use the notation $(\lambda_{0,k})_{k\in I}$ introduced before to denote the sequence of eigenvalues of A.

Proposition 1.5.8 *If (1.24) holds, then the system of generalized eigenvectors of $\mathcal{A}$ forms a Riesz basis of $\mathcal{H}$.*

Proof. For $k \in \mathbb{N}$ large enough, an eigenvector ϕ_k (respectively $\phi_{0,k}$) of $\mathcal{A}$ associated with the eigenvalue λ_k (respectively $\lambda_{0,k}$) whose imaginary $\Im(\lambda_k)$ (respectively $\Im(\lambda_{0,k})$) lies between $n_k\pi$ and $(n_k+1)\pi$ for some integer n_k and whose norm $\|\phi_k\| \sim 1$, is given by

$$\phi_k = \frac{1}{\lambda_k}\big(\sinh(\lambda_k x), \lambda_k \sinh(\lambda_k x), \lambda_k \sinh\lambda_k(\lambda_k I - B)^{-1}C\big).$$

with

$$\frac{1}{\lambda_k} = \frac{1}{in_k\pi} + o(\frac{1}{n_k}) \sim \frac{1}{i\sqrt{N(\mu_{n_k})}} \sim \frac{1}{\lambda_{0,k}}.$$

Now, we examine for k large enough the following difference,

$$\begin{aligned}
\|\phi_k - \phi_{0,k}\|^2 &= \int_0^1 |\frac{\lambda_{k,0}}{i\sqrt{(N(\mu_{n_k}))}}\cosh(\lambda_{k,0}x) - \cosh(\lambda_k x)|^2 dx \\
&+ \int_0^1 |\frac{\lambda_{k,0}}{i\sqrt{(N(\mu_{n_k}))}}\sinh(\lambda_{k,0}x) - \sinh(\lambda_k x)|^2 dx \\
&+ \int_0^1 \|\frac{1}{i\sqrt{(N(\mu_{n_k}))}}\sum_{n=0}^{\infty}\frac{B_0^n}{\lambda_{k,0}^n}C\sinh(\lambda_{k,0}) - \frac{1}{\lambda_k}\sum_{n=0}^{\infty}\frac{B^n}{\lambda_k^n}C\sinh(\lambda_k)\|^2_{\mathbb{C}^n}dx.
\end{aligned}$$

Indeed, we have

$$|\frac{\lambda_{k,0}}{i\sqrt{(N(\mu_{n_k}))}}\cosh(\lambda_{k,0}x) - \cosh(\lambda_k x)| \lesssim \frac{1}{n_k}.$$

Similarly, we obtain

$$|\frac{\lambda_{k,0}}{i\sqrt{(N(\mu_{n_k}))}}\sinh(\lambda_{k,0}x) - \sinh(\lambda_k x)|^2 \lesssim \frac{1}{n_k^2}.$$

Using the fact that $\Re(\lambda_k) \to 0, \text{ as } k \to \infty$, then for k large enough, we get

$$\|\frac{1}{i\sqrt{(N(\mu_{n_k}))}}\sum_{n=0}^{\infty}\frac{B_0^n}{\lambda_{k,0}^n}C\sinh(\lambda_{k,0}) - \frac{1}{\lambda_k}\sum_{n=0}^{\infty}\frac{B^n}{\lambda_k^n}C\sinh(\lambda_k)\|^2_{\mathbb{C}^n} \leq \frac{1}{n_k^2}.$$

We conclude that

$$\sum_{k\in I}\|\phi_k - \phi_{0,k}\|^2_{\mathcal{H}} < +\infty,$$

which implies that the conditions of Bari's theorem hold because of (1.24) (see Remark 1.5.5). ■

Definition 1.5.9 *For all $u_0 \in D(\mathcal{A})$ define $\omega(u_0)$ by*

$$\omega(u_0) = \sup\{\alpha \in \mathbb{R} : E(t) = \frac{1}{2}\|u(t)\|^2 \lesssim \frac{1}{t^\alpha}\}.$$

A decay rate is said to be optimal if it is equal to the minimum of $\omega(u_0)$ over all values of $u_0 \in D(\mathcal{A})$. Our aim is thus to find $\inf_{u_0\in D(\mathcal{A})} \omega(u_0)$. We recall the following Lemma (see [34, 53]), which will be used in the proof of optimality.

Lemma 1.5.10 *Consider a C_0-semigroup $T(t)$ acting on a (real or complex) Hilbert space $\mathcal{H}$ with infinitesimal generator $\mathcal{A}$. Assume the following*

(i) For $k \in \mathbb{N}^*$, the eigenvalue λ_k of $\mathcal{A}$ is of the form $\lambda_k = -\sigma_k + i\tau_k$ with $\sigma_k \geq \frac{c_1}{k^\delta}$, where $c_1 > 0$ and $\delta > 0$ are independent of k.

(ii) The eigenvectors $\phi_k, k \geq 1$ associated with the eigenvalue λ_k form a Riesz basis of $\mathcal{H}$.

(iii) Let $u_0 \in \mathcal{H}$ be such that

$$u_0 = \sum_{k\geq 1} a_k \phi_k, |a_k| \leq \frac{c_2}{k^q}, c_2 > 0, q > \frac{1}{2}.$$

Then there exists a constant $c > 0$ depending on u_0 such that $\|T(t)u_0\|_{\mathcal{H}} \leq \frac{c}{t^{(q-\frac{1}{2})/\delta}}, \forall t > 0$.

Remark 1.5.11 *Note that if $\sigma_k \sim \frac{1}{k^\delta}$ and $|a_k| \sim \frac{1}{k^q}$ then the equivalence*

$$\|T(t)u_0\|_{\mathcal{H}} \sim \frac{1}{t^{(q-\frac{1}{2})/\delta}}, \forall t > 0$$

holds.

Proposition 1.5.12 *Let the assumptions of Theorem 1.4.15 together with (1.24) of Remark 1.5.5 be satisfied. If $\Re(\lambda_k) \sim -\frac{1}{k^\delta}$, with $\delta \geq 2(m+1)$, then*

$$\inf_{u_0 \in D(\mathcal{A})} \omega(u_0) = \frac{1}{m+1}.$$

Proof. Let $\varepsilon > 0$ be given and let k_0 be large enough so that λ_k is algebraically simple for all $k \geq k_0$. Set $u_0^\varepsilon = \sum_{k\geq k_0} \frac{1}{k^q}\phi_k$, with $q = \frac{\delta}{2(m+1)} + \frac{1}{2} + \frac{\delta\varepsilon}{2}$. As $2(q-1) > 1$, $u_0^\varepsilon \in D(\mathcal{A})$ and $\|\mathcal{A}u_0^\varepsilon\|^2 \sim \sum_{k\geq k_0} \frac{1}{k^{2(q-1)}} < +\infty$. Moreover, due to Proposition 1.5.8 the system $(\phi_k)_{k\in I}$ forms a Riesz basis of $\mathcal{H}$. Using Remark 1.5.11, we get

$$\|u(t)\| \sim \frac{1}{t^{\frac{(q-\frac{1}{2})}{\delta}}} = \frac{1}{t^{\frac{1}{2(m+1)}+\frac{\varepsilon}{2}}}.$$

We deduce that $E(t) \sim \frac{1}{t^{\frac{1}{(m+1)}+\varepsilon}}$, and therefore

$$\frac{1}{m+1} \leq \inf_{u_0\in D(\mathcal{A})} \omega(u_0) \leq \frac{1}{m+1} + \varepsilon, \quad \forall \varepsilon > 0.$$

Hence $\inf_{u_0\in D(\mathcal{A})} \omega(u_0) = \frac{1}{m+1}$. ■

Corollary 1.5.13 *If $m = 0$ we obtain optimal polynomial energy decay given by*

$$E(t) \leq \frac{c}{t}\|\mathcal{A}u_0\|^2_{\mathcal{H}}.$$

Proof. Using Theorem 1.4.15, the solution of system (1.1) satisfies the energy estimate given by $E(t) \leq \frac{c}{t}\|\mathcal{A}u_0\|^2_{\mathcal{H}}$. Since $PC \neq 0$ and $(-RC, C) \gtrsim \alpha\|PC\|^2$ for some $\alpha > 0$, then $\Re(\lambda_k) \sim -\frac{1}{k^2}$ and the optimality is thus obtained by applying Proposition 1.5.12 with $\delta = 2$. ■

We would like now to investigate the optimality of the energy decay in the case $m = 1$.

Corollary 1.5.14 *If $m = 1$, $(RC, C) = 0$ and $\Im(B^2C, C) = 0$ then the polynomial energy decay rate is optimal.*

Proof. By straightforward computations, we get

$$\lambda = i(k\pi + \frac{\pi}{2}) - \frac{\|C\|^2}{\lambda} - \frac{(BC,C)}{\lambda^2} - \frac{\frac{\|C\|^6}{3} + (B^2C,C)}{\lambda^3} - \frac{\|C\|^4(BC,C) + (B^3C,C)}{\lambda^4} + o\left(\frac{1}{\lambda^4}\right).$$

Then by further calculation, as $(RC, C) = 0$ we obtain

$$\Re(\lambda) = \frac{\Im(B^2C,C)}{k^3\pi^3} - \frac{3\Im(B^2C,C)}{2k^4\pi^3} - \frac{\Re(B^3C,C)}{k^4\pi^4} + o\left(\frac{1}{k^4}\right).$$

Moreover, since $\Im(B^2C, C) = \Im((RB_0 + B_0R)C, C) = 0$, we get

$$\Re\lambda = -\frac{\Re(B^3C,C)}{k^4\pi^4} + o\left(\frac{1}{k^4}\right),$$

and the optimality follows. ■

Remark 1.5.15 *In the case $m = 1$, if $B \in M_n(\mathbb{R}), M \in M_n(\mathbb{R})$ and $C \in \mathbb{R}^n$ then the sufficient conditions to obtain optimality reduce to the first condition $(RC, C) = 0$.*

1.6 Examples

In this section, we present some examples and applications, in which we obtain polynomial stability and check the optimality of the energy decay rate.

1.6.1 Example 1.

Let us consider the following system (P_{b_0,b_1}) given by:

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) &= 0, \quad 0 < x < 1, t > 0, \\ y(0,t) &= 0, \quad t > 0, \\ y_x(1,t) + \eta_t(t) &= 0, \quad t > 0, \\ \eta_{tt}(t) + b_1\eta_t(t) + b_0\eta(t) - y_t(1,t) &= 0, \quad t > 0, \end{cases}$$

where y represents the transversal displacement of the vibrating string and η denotes the dynamical control variable. Here b_0 and b_1 are positive constants.
This system is nothing but the system considered in [16, 38] with a scalar variable instead of a vectorial one in $\mathbb{R}^3$. In [38], the authors obtained polynomial stability using a multiplier method but no optimality of the polynomial decay was proven. By our study, the optimal polynomial decay is obtained.

In this case, we have

$$n = 2, \quad B = \begin{pmatrix} 0 & 1 \\ -b_0 & -b_1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The inner product considered on $\mathbb{C}^2$ is given by

$$\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right)_{\mathbb{C}^2} = b_0 x\bar{x}_1 + y\bar{y}_1, \text{ or } M = \begin{pmatrix} b_0 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$B^* = \begin{pmatrix} 0 & -1 \\ b_0 & -b_1 \end{pmatrix}, B_0 = \begin{pmatrix} 0 & 1 \\ -b_0 & 0 \end{pmatrix}, R = \begin{pmatrix} 0 & 0 \\ 0 & -b_1 \end{pmatrix}.$$

The energy space is

$$\mathcal{H} = V \times L^2(0,1) \times \mathbb{C}^2,$$

endowed with the following inner product

$$((y,z,\eta,\kappa),(y_1,z_1,\eta_1,\kappa_1)) = \int_0^1 y_x \bar{y}_{1x} dx + \int_0^1 z\bar{z}_1 dx + b_0 \eta\bar{\eta}_1 + \kappa\bar{\kappa}_1.$$

Define

$$D(\mathcal{A}) = \{(y,z,\eta,\kappa) \in \mathcal{H}^2(0,1) \cap V \times V \times \mathbb{C}^2 : y_x(1) = -\kappa\},$$

and

$$\mathcal{A}\begin{pmatrix} y \\ z \\ \eta \\ \kappa \end{pmatrix} = \begin{pmatrix} z \\ y_{xx} \\ \kappa \\ z(1) - b_0\eta - b_1\kappa \end{pmatrix}.$$

B is Hurwitz, since

$$\det(\lambda I - B) = \lambda^2 + b_1\lambda + b_0$$

has no pure imaginary roots. Moreover, we have

$$\Re(B\eta,\eta) \leq 0 \text{ and } |\det(izI - B)|^2 \Re\left((izI - B)^{-1}C, C\right) = b_1 z^2 > 0, \forall z \in \mathbb{R}^*.$$

Due to Proposition 1.2.1 and Corollary 1.3.10, we deduce the well-posedness of the system and the asymptotic stability of its energy.
In addition, the conditions of Lemma 1.4.8 are satisfied. In fact, we have

$$\det(\lambda I - B_0) = \lambda^2 + b_0, \sigma(B_0) = \{\pm i\sqrt{b_0}\} \text{ and } \operatorname{adj}(\lambda I - B_0)C = \begin{pmatrix} 1 \\ \lambda \end{pmatrix},$$

and the spaces $\ker R$ and W are given by span $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ and span $\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ respectively, thus

$$P((\lambda I - B_0)^{-1}C) \neq 0, \forall \lambda \in \sigma(A) \setminus \sigma(B_0).$$

To check the rest of the conditions of the lemma, we distinguish two cases:

Case 1. If $\mu^2 = b_0 \neq k^2\pi^2$ for all $k \in \mathbb{Z}$, then

$$C \notin \ker(i\mu I - B_0)^{\perp}.$$

Indeed, computing

$$\left(\begin{pmatrix} 1 \\ i\mu \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = i\mu \neq 0,$$

implies that $C \notin \ker(i\mu I - B_0)^{\perp}$, as we have $\eta \in \ker(i\mu I - B_0)$ if and only if $\eta = (1, i\mu)^{\top}$ up to a nonzero constant. So by Remark 1.4.9, $\sigma(A) \cap \sigma(B_0) \subset \{\pm ik\pi : k \in \mathbb{Z}\}$. In fact, due to Proposition 1.3.6, $\sigma(A) \cap \sigma(B_0) \subset \{\pm ik\pi : k \in \mathbb{Z}\} \cap \sigma(B_0)$, thus $\sigma(A) \cap \sigma(B_0) = \phi$.

Case 2. If $b_0 = k^2\pi^2$ for some $k \in \mathbb{N}^*$, then $\sigma(B_0) = \{\pm ik\pi\}$. Computing the associated eigenvectors we get

$$\eta_{\pm k\pi} = \begin{pmatrix} 1 \\ \pm ik\pi \end{pmatrix} \text{ and } P(\eta_{\pm k\pi}) = \begin{pmatrix} 0 \\ \pm ik\pi \end{pmatrix} \neq 0.$$

As (1.21) holds, then applying Theorem 1.4.15 for $m = 0$ (since $PC \neq 0$), we deduce that the energy of the system satisfies the following polynomial decay

$$E(t) \leq \frac{1}{1+t}\|u_0\|^2_{D(\mathcal{A})}.$$

It can be easily checked that $C \notin \ker(\bar{\lambda}I - B^*)^{\perp} \cup \ker(\lambda I - B)^{\perp}$, for all $\lambda \in \sigma(B)$, which allows us to deduce that condition (1.24) of Remark 1.5.5 is satisfied, as we already checked that $C \notin \ker(i\mu I - B_0)^{\perp}$. Due to Corollary 1.5.13, we conclude the optimality of the energy decay.

1.6.2 Example 2.

Considering the following boundary conditions at $x = 1$ in system (1.1)

$$\begin{cases} y_x(1,t) + b_0\eta(t) & = & 0, & t > 0, \\ \eta_t(t) - \kappa(t) - y_t(1,t) & = & 0, & t > 0, \\ \kappa_t(t) + b_0\eta(t) + b_1\kappa(t) & = & 0, & t > 0, \end{cases}$$

we get a system of the form (1.1) which is obtained by replacing $C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in the first example by $C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and keeping B, B^*, B_0, and R as before.

Moreover, for all $z \in \mathbb{R}$ we have

$$|\det(izI - B)|^2\Re\left((izI - B)^{-1}C, C\right) = b_1 b_0^2 > 0,$$

thus using Corollary 1.3.10, we deduce the asymptotic stability of its energy.

In addition, to verify the conditions of Lemma 1.4.8, it is enough to remark that

$$P(\mathrm{adj}(\lambda I - B_0)C) = P\begin{pmatrix} \lambda \\ -b_0 \end{pmatrix} = \begin{pmatrix} 0 \\ -b_0 \end{pmatrix} \neq 0,$$

and as the discussion of the case $b_0 = k^2\pi^2$ of the first example remains unchanged, to check that $\sigma(A) \cap \sigma(B_0) \subset \{\pm ik\pi : k \in \mathbb{Z}\}$, we just remark that

$$(-i\mu I + B_0)C = \begin{pmatrix} -i\mu \\ -b_0 \end{pmatrix} \neq 0, \text{ and } \left(\begin{pmatrix} 1 \\ i\mu \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)_{\mathbb{C}^n} = b_0 \neq 0.$$

Applying Theorem 1.4.15 for $m = 1$ (since $PC = 0$ and $P(B_0C) \neq 0$), we deduce that the energy of the system satisfies the following polynomial decay

$$E(t) \leq \frac{1}{\sqrt{1+t}}\|u_0\|^2_{D(\mathcal{A})}.$$

Moreover, by the first example condition (1.24) holds, thus the decay is optimal by Remark 1.5.15, since $RC = 0$.

1.6.3 Example 3.

Consider the following system in which the dynamical boundary control involves a third order differential equation of the dynamical variable η,

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) & = & 0, & 0<x<1, t>0, \\ y(0,t) & = & 0, & t>0, \\ y_x(1,t) + 2\eta(t) + 2\eta_t(t) + \eta_{tt}(t) & = & 0, & t>0, \\ \eta_{ttt}(t) + 2\eta_{tt}(t) + 3\eta_t(t) + \eta(t) - y_t(1,t) & = & 0, & t>0. \end{cases}$$

By introducing the following variables $\eta_1 = 2\eta + 2\eta_t + \eta_{tt}, \eta_2 = -\eta - \eta_t$, $\eta_3 = \eta$, the last two equations can be rewritten in the form

$$\begin{cases} y_x(1,t) + (\delta(t), C) & = & 0, & t>0, \\ \delta_t(t) - B\delta(t) - Cy_t(1) & = & 0, & t>0, \end{cases}$$

where $\delta = (\eta_1, \eta_2, \eta_3)^\top$, and $(\cdot,\cdot)$ denotes the usual inner product defined on $\mathbb{C}^3$, and the matrices B and C are given by:

$$B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & -1 & -1 \end{pmatrix}, C = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We have $\Re(B\delta, \delta) = -|\eta_2|^2 - |\eta_3|^2 \leq 0$ and $\det(\lambda I - B) = \lambda^3 + 2\lambda^2 + 3\lambda + 1$, thus B has no pure imaginary eigenvalues. We also have $|\det(izI - B)|^2 \Re((izI-B)^{-1}C, C) = z^2 + 2 > 0,$ for all $z \in \mathbb{R}^*$. Using Proposition 1.2.1 and Corollary 1.3.10, the solution of the proposed system exists and unique and its energy is asymptotically stable. We proceed by checking the conditions of Lemma 1.4.8. Computing the characteristic equation of B_0, we get

$$\det(\lambda I - B_0) = \lambda(\lambda^2 + 2) \text{ and } \sigma(B_0) = \{0, \pm i\sqrt{2}\}.$$

For $\lambda = \pm i\sqrt{2}$, we have $\ker(\lambda I - B_0) = \text{span}\{(-\frac{\lambda}{2}, 1, \frac{\lambda}{2})^\top\}$, it follows that $(C, (-\frac{\lambda}{2}, 1, \frac{\lambda}{2})^\top) = -\frac{\lambda}{2}$, which is nonzero thus $C \notin \ker(\lambda I - B_0)^\perp$. Computing W and $\text{adj}(\lambda I - B_0)C$, we obtain

$$W = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad \text{adj}(\lambda I - B_0)C = \begin{pmatrix} \lambda^2 + 1 \\ -\lambda \\ 1 \end{pmatrix},$$

then $P((\lambda I - B_0)^{-1}C) \neq 0$ for all $\lambda \in \sigma(A) \setminus \sigma(B_0)$. The eigenvector of B_0 associated with $\lambda = 0$ and its projection on W are given by

$$\eta = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad P\eta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \neq 0.$$

Substituting m by 1 in Theorem 1.4.15 (since $PC = 0$ and $P(B_0C) \neq 0$), we deduce that the energy of the solution of (1.26) fulfills the following polynomial estimate

$$E(t) \leq \frac{1}{\sqrt{1+t}} \|\mathcal{A}u_0\|^2_{D(\mathcal{A})}.$$

Verifying condition (1.24) and noting that B and C have real components and $(RC, C) = 0$, we conclude the optimality by Remark 1.5.15.

1.6.4 Example 4.

Consider the following system given by:

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) & = & 0, & 0 < x < 1, t > 0, \\ y(0,t) & = & 0, & t > 0, \\ y_x(1,t) - \eta(t) - \eta_t(t) - \eta_{tt}(t) & = & 0, & t > 0, \\ \eta_{ttt}(t) + \eta_{tt}(t) + 2\eta_t(t) + \eta(t) + y_t(1,t) & = & 0, & t > 0. \end{cases}$$

Choosing

$$\eta_1 = -\eta - \eta_t, \eta_2 = -\eta - \eta_t - \eta_{tt}, \eta_3 = \eta,$$

we get a system in the form of system (1.1) with

$$B = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}, B_0 = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, C = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

As $\det(izI - B) = -iz^3 - z^2 + 2iz + 1 \neq 0, \forall z \in \mathbb{R}$ and $\Re(B\eta, \eta) \leq 0$, where $(\cdot, \cdot)$ denotes the usual inner product defined on $\mathbb{C}^3$, we deduce that B is Hurwitz. By a straight forward calculation we have

$$|\det(izI - B)|^2 \Re((izI - B)^{-1}C, C) = 1 > 0, \forall z \in \mathbb{R}$$

we deduce by Proposition 1.2.1 the existence and uniqueness of the solution of the system and the asymptotic stability of its energy follows from Corollary 1.3.10. We also have

$$\mathrm{adj}(\lambda I - B_0)C = \begin{pmatrix} \lambda \\ \lambda^2 + 1 \\ -1 \end{pmatrix},$$

knowing that the space W is spanned by $(0,0,1)^\top$ we deduce $P((\lambda I - B_0)^{-1}C) \neq 0$, for every $\lambda \in \sigma(A) \setminus \sigma(B_0)$. Moreover, the characteristic equation of B_0 is given by $\det(\lambda I - B_0) = \lambda(\lambda^2 + 2)$, thus the eigenvalues of B_0 are $0, \pm i\sqrt{2}$. The eigenvector associated with zero is given by $(0,1,-1)^\top$, whose projection on W is nonzero and $\ker(\lambda I - B_0)$ is spanned by $(\frac{2}{\lambda}, 1, 1)^\top$ for $\lambda = \pm i\sqrt{2}$, then $C \notin \ker(\lambda I - B_0)^\perp$. We deduce that the conditions of Lemma 1.4.8 holds and we therefore get the following estimate

$$E(t) \leq \frac{1}{(1+t)^{\frac{1}{3}}} \|u_0\|^2_{D(\mathcal{A})}.$$

by simply replacing m by 2 in Theorem 1.4.15, as $PC = 0, P(B_0C) = 0$, and $P(B_0^2C) \neq 0$. Moreover, computing the asymptotic expansion of $\lambda \in \sigma(\mathcal{A})$ for a modulus large enough with $\Im(\lambda) \in (k\pi, (k+1)\pi)$, we get

$$\lambda = i(k\pi + \frac{\pi}{2}) - \frac{1}{\lambda} + \frac{2}{3\lambda^3} - \frac{6}{5\lambda^5} + \frac{1}{\lambda^6} + o\left(\frac{1}{\lambda^6}\right),$$

then $\Re(\lambda) = -\frac{1}{k^6\pi^6} + o\left(\frac{1}{k^6}\right)$, in addition $C \notin \ker(\bar{\lambda}I - B^*)^\perp \cup \ker(\lambda I - B)^\perp$, thus the conditions of Proposition 1.5.12 holds and the optimality of the energy decay follows.

1.6.5 Example 5.

Consider the following system given by:

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) &= 0, \quad 0 < x < 1, t > 0, \\ y(0,t) &= 0, \quad t > 0, \\ y_x(1,t) - b_1\eta(t) - \eta_t(t) + \kappa(t) &= 0, \quad t > 0, \\ \eta_{tt}(t) + b_1\eta_t(t) + b_0\eta(t) + b_0 y_t(1,t) &= 0, \quad t > 0, \\ \kappa_t(t) + b_2\kappa(t) - y_t(1,t) &= 0, \quad t > 0. \end{cases}$$

with b_0, b_1, b_2 positive constants. Choosing

$$\eta_1 = -\frac{\eta_t + b_1\eta}{b_0}, \eta_2 = \eta, \eta_3 = \kappa,$$

we get a system in the form of (1.1) with

$$B = \begin{pmatrix} 0 & 1 & 0 \\ -b_0 & -b_1 & 0 \\ 0 & 0 & -b_2 \end{pmatrix}, C = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

In addition,

$$M = \begin{pmatrix} b_0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B_0 = \begin{pmatrix} 0 & 1 & 0 \\ -b_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -b_1 & 0 \\ 0 & 0 & -b_2 \end{pmatrix}, \operatorname{adj}(\lambda I - B_0)C = \begin{pmatrix} \lambda^2 \\ -b_0\lambda \\ \lambda^2 + b_0 \end{pmatrix}.$$

It is easy to check that $\Re(B\eta, \eta) \leq 0$, and since $\det(\lambda I - B) = (\lambda + b_2)(\lambda^2 + b_1\lambda + b_0)$, then B is Hurwitz. We also have

$$|\det(izI - B)|^2 \Re\left((izI - B)^{-1}C, C\right) = b_2 z^4 + (b_0^2 b_1 + b_2 b_1^2 - 2b_0 b_2)z^2 + b_0^2 b_2(b_1 b_2 + 1),$$

which is positive for all $z \in \mathbb{R}$, we thus obtain an asymptotically stable system. The space W is given by $\operatorname{span}\{(0,1,0)^\top, (0,0,1)^\top\}$, then

$$P(\operatorname{adj}(\lambda I - B_0)C) = (0, -b_0\lambda, \lambda^2 + b_0)^\top \neq 0, \quad \forall \lambda \notin \sigma(B_0).$$

The spectrum of B_0 is given by $\sigma(B_0) = \{0, \pm i\sqrt{b_0}\}$. The eigenvector of B_0 associated with 0 is equal to $(0,0,1)^\top$ up to a nonzero constant, thus $P(0,0,1)^\top \neq 0$. For $\lambda = \pm i\mu = \pm i\sqrt{b_0}$, we have

$$\eta_\mu = (1, \pm i\mu, 0)^\top \text{ and } P\eta_\mu \neq 0.$$

Moreover, it can be easily checked that $C \notin \ker(\lambda I - B_0)^\perp$ for all $\lambda \in \sigma(B_0)$. Hence Theorem 1.4.15 applied for $m = 0$ (as $PC \neq 0$), gives the following energy decay estimate

$$E(t) \leq \frac{1}{1+t}\|\mathcal{A}u_0\|_{\mathcal{H}}^2.$$

Finally, we may easily verify that for all geometrically simple $\lambda \in \sigma(B)$ we have $C \notin \ker(\bar{\lambda}I - B^*)^\perp \cup \ker(\lambda I - B)^\perp$. Consequently, if B has geometrically simple eigenvalues (or equivalently if $b_2^2 + b_0 \neq b_1 b_2$), then the optimality of the polynomial decay rate can be deduced from Corollary 1.5.13.

2 The multidimensional wave equation with generalized acoustic boundary conditions

2.1 Introduction

Denote by Ω a bounded open connected set of $\mathbb{R}^d$, $d \geq 1$, with a Lipschitz boundary $\partial\Omega = \Gamma$ assumed to be divided into two disjoint parts,

$$\Gamma = \Gamma_0 \cup \Gamma_1,$$

where Γ_0 is assumed to be closed with a nonempty interior and Γ_1 relatively open in Γ which could be possibly empty. We further assume that $\Gamma_0 \cap \bar{\Gamma}_1$ is of class C^1 in the sense explained later.

For $n \in \mathbb{N}^*$, we further fix $C \in C^{0,1}(\Gamma_0, \mathbb{C}^n)$ and a matrix valued function $B \in C^{0,1}(\Gamma_0, M_n(\mathbb{C}))$ and for every $x \in \Gamma_0$, an inner product $(\cdot,\cdot)_x$ in $\mathbb{C}^n$ such that

$$\Re(B(x)\cdot,\cdot)_x \leq 0. \tag{2.1}$$

For every $x \in \Gamma_0$, let $M(x) \in M_n(\mathbb{C})$ be the Hermitian positive-definite matrix associated with this inner product, i.e.

$$(\delta_1, \delta_2)_x = \bar{\delta}_2^\intercal M(x)\delta_1, \forall \delta_1, \delta_2 \in \mathbb{C}^n.$$

From now on we further assume that M is Lipschitz continuous on Γ_0. For the sake of brevity, if there is no confusion we use the notation $(\cdot,\cdot)$ to denote $(\cdot,\cdot)_x$. The associated norm is denoted by $\|\cdot\|$.

We consider the following evolution problem with a Dirichlet boundary condition on Γ_1 and a dynamical control on Γ_0, described as follows:

$$\begin{cases} y_{tt}(x,t) - \Delta y(x,t) = 0 & , x \in \Omega, t > 0, \\ y(x,t) = 0 & , x \in \Gamma_1, t > 0, \\ \frac{\partial y}{\partial \nu}(x,t) = (\delta(x,t), C) & , x \in \Gamma_0, t > 0, \\ \delta_t(x,t) = B\delta(x,t) - Cy_t(x,t) & , x \in \Gamma_0, t > 0, \end{cases} \tag{2.2}$$

with the following initial conditions:

$$\begin{cases} y(x,0) = y_0(x), y_t(x,0) = y_1(x), \quad x \in \Omega, \\ \delta(x,0) = \delta_0(x), x \in \Gamma_0, \end{cases} \tag{2.3}$$

where y is a complex valued function (representing the transverse displacement in the case $\Omega \subset \mathbb{R}$ and the potential velocity in the case $\Omega \subset \mathbb{R}^d$, with $d \geq 2$) and δ denotes the dynamical control variable.

In a first step, we try to find sufficient conditions that guarantee the strong stability of the system. Here, as the domain of the associated operator is not compactly embedded into the natural energy space, we may expect that its spectrum is not only made of eigenvalues. We prove such a result in our general setting but since Dirichlet boundary conditions are imposed on a part of the boundary, we were not able to use the single-layer potential technique of [16] and instead we use a Fredholm alternative technique. Finally, similar assumptions on B and C as in the one-dimensional case allow us to show that the associated operator has no eigenvalues on the imaginary axis, hence we obtain the strong stability by using Arendt-Batty theorem (see [10] and Theorem 1.3.1). In dimension one, using the compact perturbation result of Russell [46], the dissipative system (2.2) is not uniformly stable (see [2] and [37, Rk 2]). In dimension larger than 2, this argument cannot be used, but nevertheless by using the spectral properties of the Laplace operator with specific Robin boundary conditions on Γ_0, we will show that the resolvent of the associated operator is not uniformly bounded on the imaginary axis and by the frequency domain approach [23, 26, 43], we will conclude that our system is not uniformly stable. Hence we are interested in proving a weaker decay of the energy. More precisely, we will give sufficient conditions on Γ_0, B and C that yield the polynomial decay of the energy of our system (for initial data in the domain of the associated operator). A first approach is to use a multiplier method (as in [38, 50, 53]) but this approach requires a quite strong geometrical assumption on Γ_0. Hence we alternatively use the frequency domain approach from [19]. In this case, we find an appropriate bound for the resolvent on the imaginary axis by using the exponential or polynomial decay of the wave equation with the standard damping

$$\frac{\partial y}{\partial \nu}(x,t) = -y_t \text{ on } \Gamma_0,$$

and an assumption on the behavior of $\Re((isI - B)^{-1}C, C)$ for all real number s whose modulus is large enough. This leads to quite weaker geometrical assumption on Γ_0 due to the results from [12, §5] or [28, 29] for instance. In particular, with this second approach as Γ_1 can be empty, we significantly improve results from [16] and [38].

The chapter is organized as follows. The second section deals with the well-posedness of the problem obtained by using semigroup theory. Section 2.3 is devoted to the analysis of the spectrum of the associated operator that is characterized by using a Fredholm alternative technique. The strong stability of the system is studied in section 2.4 by using Arendt-Batty theorem. In section 2.5, we show that the resolvent of the operator is not uniformly bounded on the imaginary axis and deduce that our system is not uniformly stable. Section 2.6 is devoted to the proof of the polynomial decay of our system by using the frequency domain approach, while in section 2.8 we prove a similar polynomial decay by using the multiplier method. We shortly look for the case Γ_1 empty in section 2.7. Finally some particular examples illustrating our general framework are presented in section 2.9.

In the whole chapter, we assume that Γ_1 is nonempty (without any specification), the case Γ_1 empty is only supposed in section 2.7 (and in section 2.9). The case $d = 1$ was discussed in the first chapter, we assume that $d > 1$ throughout the remainder of the chapter.

2.2 Well-posedness results

As usual, to prove existence result for system (2.2) we use a reduction of order argument. Hence, we define

$$V = \{y \in H^1(\Omega) : y = 0 \text{ on } \Gamma_1\} = H^1_{\Gamma_1}(\Omega),$$

that is a Hilbert space endowed with the following inner product $(y_1, y_2)_V = \int_0^1 \nabla y_1 \nabla \bar{y}_2 dx$, and norm $\|y\|_V = (y, y)_V^{\frac{1}{2}}$. The energy space is then

$$\mathcal{H} = V \times L^2(\Omega) \times (L^2(\Gamma_0))^n,$$

endowed with the following inner product,

$$((y, z, \delta), (y_1, z_1, \delta_1))_{\mathcal{H}} = \int_\Omega \nabla y \nabla \bar{y}_1 dx + \int_\Omega z \bar{z}_1 dx + \int_{\Gamma_0} (\delta, \delta_1) dx. \tag{2.4}$$

We define the unbounded operator associated with the evolution problem by $(\mathcal{A}, D(\mathcal{A}))$,

$$\mathcal{A}U = \begin{pmatrix} z \\ \Delta y \\ B\delta - C\gamma_0 z \end{pmatrix}, U = \begin{pmatrix} y \\ z \\ \delta \end{pmatrix} \in D(\mathcal{A}),$$

where $D(\mathcal{A}) = \{U \in \mathcal{H} : \Delta y \in L^2(\Omega), z \in V, \frac{\partial y}{\partial \nu} = C^\intercal M\delta \text{ on } \Gamma_0\}$, in the last component $\gamma_0 z$ is the trace of z on Γ_0 and the boundary condition

$$\frac{\partial y}{\partial \nu} = (\delta, C) \text{ on } \Gamma_0, \tag{2.5}$$

is to be viewed in the following weak sense (see [25]):

$$\int_\Omega \Delta y \varphi dx + \int_\Omega \nabla y \nabla \varphi dx = \int_{\Gamma_0} (\delta, C) \gamma_0 \varphi ds, \forall \varphi \in V. \tag{2.6}$$

If y and δ are solution of system (2.2) and are sufficiently smooth, we easily check that $U = (y, z, \delta)^\top$ is solution of the Cauchy problem

$$U_t = \mathcal{A}U, \quad U(0) = U_0 \tag{2.7}$$

with $U_0 = (y_0, y_1, \delta_0)^\top$.

The energy of our system (2.2) (or (2.7)) is then naturally defined by

$$E_0(t) = \frac{1}{2}\|U(t)\|^2_{\mathcal{H}} = \frac{1}{2}\left(\int_\Omega |\nabla y|^2 dx + \int_\Omega |y_t|^2 dx + \int_{\Gamma_0} \|\delta\|^2 ds\right), \tag{2.8}$$

for $U(t) = (y, y_t, \delta) \in V \times L^2(\Omega) \times (L^2(\Gamma_0))^n$ solution of (2.7).

Hence, we proceed by proving that $\mathcal{A}$ is m-dissipative.

Proposition 2.2.1 *The operator $\mathcal{A}$ is m-dissipative.*

Proof. Let $U = \begin{pmatrix} y \\ z \\ \delta \end{pmatrix} \in D(\mathcal{A})$. Then as $z \in V$, using (2.6) we get:

$$\begin{aligned}\left(\mathcal{A}\begin{pmatrix} y \\ z \\ \delta \end{pmatrix}, \begin{pmatrix} y \\ z \\ \delta \end{pmatrix}\right)_{\mathcal{H}} &= \int_\Omega \nabla z \nabla \bar{y} dx + \int_\Omega \Delta y \bar{z} dx + \int_{\Gamma_0} (B\delta - C\gamma_0 z, \delta)_{\mathbb{R}^n} ds \\ &= \int_\Omega \nabla z \nabla \bar{y} dx - \int_\Omega \nabla y \nabla \bar{z} dx + \int_{\Gamma_0} (\delta, C)\gamma_0 \bar{z} ds + \int_{\Gamma_0} (B\delta - C\gamma_0 z, \delta)_{\mathbb{R}^n} ds.\end{aligned}$$

Consequently,

$$\Re(\mathcal{A}U, U) = \int_{\Gamma_0} \Re(B\delta, \delta)_{\mathbb{R}^n} ds \le 0, \tag{2.9}$$

and thus $\mathcal{A}$ is dissipative.

We would like to show that there exists $\lambda > 0$ such that $\lambda I - \mathcal{A}$ is surjective. Let $\lambda > 0$ be given. For $F = (y_1, z_1, \delta_1)^\top \in \mathcal{H}$, we look for $U = (y, z, \delta)^\top \in D(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A})U = F,$$

or equivalently

$$\begin{cases} \lambda y - z = y_1, \\ \lambda z - \Delta y = z_1, \\ (\lambda I - B)\delta + C\gamma_0 z = \delta_1. \end{cases} \tag{2.10}$$

Assume that such a $(y, z, \delta)^\top \in D(\mathcal{A})$ exists, then $z = \lambda y - y_1$, and as $\lambda \notin \sigma(B)$, δ is given by

$$\delta = (\lambda I - B)^{-1} \left(\delta_1 + C\gamma_0 \left(y_1 - \lambda y\right)\right). \tag{2.11}$$

Hence, $y \in V$ satisfies

$$\lambda^2 y - \Delta y = z_1 + \lambda y_1, \tag{2.12}$$

and the boundary condition

$$\frac{\partial y}{\partial \nu} = (\delta, C) \text{ on } \Gamma_0.$$

We first look for an associated weak formulation of this problem on y (and then prove that it admits a unique solution using Lax-Milgram lemma). Multiplying (2.12) by a function $\varphi \in V$, integrating the obtained identity in Ω and by (2.6) (allowed since we assume that $y \in V$ exists with the property $\Delta y \in L^2(\Omega)$), we find

$$a_\lambda(y, \varphi) = L_F(\varphi), \quad \forall \varphi \in V, \tag{2.13}$$

where a_λ and L_F are given by

$$a_\lambda(y, \varphi) = \int_\Omega \lambda^2 y \bar{\varphi} dx + \int_\Omega \nabla y \nabla \bar{\varphi} dx + \int_{\Gamma_0} \lambda((\lambda I - B)^{-1} C, C)\gamma_0 y \gamma_0 \bar{\varphi} ds, \tag{2.14}$$

$$L_F(\varphi) = \int_\Omega (z_1 + \lambda y_1)\bar{\varphi} dx + \int_{\Gamma_0} \left((\lambda I - B)^{-1}(C\gamma_0 y_1 + \delta_1), C\right) \gamma_0 \bar{\varphi} ds. \tag{2.15}$$

Clearly, $\bar{L}_F$ is a linear continuous functional on V, and $\bar{a}_\lambda$ is a sesquilinear continuous form on V. Finally, a_λ is coercive on V, indeed, for any $y \in V$, we have

$$a_\lambda(y,y) = \lambda^2 \int_\Omega |y|^2 dx + \int_\Omega |\nabla y|^2 dx + \int_{\Gamma_0} ((\lambda I - B)^{-1} C, C)\lambda |\gamma_0 y|^2 ds.$$

But (2.1) implies that $\Re((\lambda I - B)^{-1}C, C) \geq 0$, since

$$\Re\left((\lambda I - B)^{-1}C, C\right) = \Re\left(u, (\lambda I - B)u\right) = \lambda\|u\|^2 - \Re(u, Bu) \geq 0,$$

with $u = (\lambda I - B)^{-1}C$. Hence, $\Re a_\lambda(y,y) \gtrsim \|y\|_V^2$, which implies that a_λ is coercive.

Applying Lax-Milgram Lemma, there exists a unique solution $y \in V$ of (2.13). In particular, taking $\varphi \in \mathcal{D}(\Omega)$ in (2.13), we get

$$\lambda^2 y - \Delta y = z_1 + \lambda y_1 \text{ in } \mathcal{D}'(\Omega). \tag{2.16}$$

We deduce that $\Delta y \in L^2(\Omega)$. Substitute $\lambda^2 y$ by $\Delta y + z_1 + \lambda y_1$ in (2.13), we obtain

$$\int_\Omega \Delta y \bar{\varphi} dx + \int_\Omega \nabla y \nabla \bar{\varphi} dx + \int_{\Gamma_0} \left((\lambda I - B)^{-1} C\lambda\gamma_0 y, C\right)\gamma_0\bar{\varphi} ds$$
$$= \int_{\Gamma_0} \left((\lambda I - B)^{-1}(\delta_1 + C\gamma_0 y_1), C\right)\gamma_0\bar{\varphi} ds.$$

Defining $\delta = (\lambda I - B)^{-1}(\delta_1 + C\gamma_0 y_1 - C\lambda\gamma_0 y) \in \left((L^2(\Gamma_0)\right)^n$, we get $\frac{\partial y}{\partial \nu} = (\delta, C)$. By defining $z = \lambda y - y_1$, we deduce the surjectivity of $\lambda I - \mathcal{A}$. ■

Remark 2.2.2 *From the previous proof, we see that if* 0 *is not an eigenvalue of* $B(x)$*, for all* $x \in \Gamma_0$*, then* $\mathcal{A}$ *is bijective and* $\mathcal{A}^{-1}$ *is bounded. The converse also holds, see Proposition 2.4.1 below.*

Since $\mathcal{A}$ is m-dissipative, then Lumer-Phillips theorem implies that $\mathcal{A}$ generates a C_0-semigroup of contractions on $\mathcal{H}$ (see for instance [42]), and allows us to state the following results.

Corollary 2.2.3 *(i) For an initial datum* $U_0 \in \mathcal{H}$ *there exists a unique solution* $U \in C([0,+\infty), \mathcal{H})$ *of (1.3). Moreover, if* $U_0 \in D(\mathcal{A})$*, then*

$$U \in C([0,+\infty), D(\mathcal{A})) \cap C^1([0,+\infty), \mathcal{H}).$$

(ii) For each $U_0 \in D(\mathcal{A})$*, the energy* $E_0(t)$ *of the solution* U *of (1.3) satisfies*

$$\frac{d}{dt}E_0(t) = \Re\int_{\Gamma_0} (B\delta, \delta) ds \leq 0,$$

and therefore the energy is non-increasing.

Proof. (i) is a direct consequence of Lumer-Phillips theorem, (ii) holds simply since

$$\frac{dE(t)}{dt} = \Re(\frac{dU(t)}{dt}, U(t)) = \Re(\mathcal{A}U(t), U(t)),$$

for all $U \in D(\mathcal{A})$. ■

Remark 2.2.4 *If* $U_0 \in D(\mathcal{A}^m)$*, then*

$$U \in C([0,+\infty), D(\mathcal{A}^m)) \cap C^1([0,+\infty), D(\mathcal{A}^{m-1})) \cap ... \cap C^m([0,+\infty), \mathcal{H}).$$

2.3 The spectrum of $\mathcal{A}$

As $D(\mathcal{A})$ is not compactly embedded into $\mathcal{H}$, we can expect that the spectrum of $\mathcal{A}$ is not only constituted of eigenvalues. This is indeed the case if Γ_1 is empty and in the setting of example 1 below as shown in [16]. Our aim is to prove such a result in our general setting. Since Dirichlet boundary conditions are imposed on a part of the boundary, we were not able to use the single-layer potential technique of [16] and we instead use a Fredholm alternative technique.

Recall that an operator T from a Hilbert space X into itself is called singular if there exists a sequence $u_n \in D(T)$ with no convergent subsequence such that $\|u_n\|_X = 1$ and $Tu_n \to 0$ in X, see [54]. According to Theorem 1.14 of [54] T is singular if and only if its kernel is infinite dimensional or its range is not closed.

Now define

$$\Sigma := \{\lambda \in \mathbb{C} : \exists x \in \Gamma_0 : \lambda I - B(x) \text{ is not invertible } \}.$$

From the continuity of B, Σ is a compact subset of $\mathbb{C}$.

We state in the following theorem some spectral properties of $\mathcal{A}$ (compare with Theorem 3.2 of [16]).

Theorem 2.3.1 *The following results hold:*

1. *If $\lambda \in \Sigma$, then $\lambda - \mathcal{A}$ is singular,*
2. *If $\lambda \notin \Sigma$, then $\lambda - \mathcal{A}$ is a Fredholm operator of index zero.*

Proof. To prove the first point, we fix $\lambda \in \Sigma$. Then there exists $x_0 \in \Gamma_0$ and $\delta \in \mathbb{C}^n$, $\delta \neq 0$ such that

$$(\lambda I - B(x_0))\delta = 0.$$

Denote by $-\Delta_m$ the positive self-adjoint operator defined by

$$D(-\Delta_m) = \{y \in V : \Delta y \in L^2(\Omega) \text{ and } \frac{\partial y}{\partial \nu} = 0 \text{ on } \Gamma_0\},$$

and

$$-\Delta_m y = -\Delta y, \forall y \in D(-\Delta_m).$$

Denote by $\{\lambda_k^2\}_{k\in\mathbb{N}^*}$ the (discrete) spectrum of $-\Delta_m$ (repeated according to their multiplicity) and let φ_k be the corresponding orthonormalized eigenvectors.

According to the Fredholm alternative, for any complex number μ we have the two following cases:

i) either $\mu \neq -\lambda_k^2$, for all $k \in \mathbb{N}^*$ and for an arbitrary $F \in V'$, there exists a unique solution of $y \in V$ of

$$\int_\Omega (\mu y \bar{w} + \nabla y \cdot \nabla \bar{w})\, dx = F(w), \forall w \in V, \tag{2.17}$$

ii) or there exists $k_0 \in \mathbb{N}^*$ such that $\mu = -\lambda_{k_0}^2$ and then for any $F \in V'$ such that

$$F(\varphi_k) = 0, \forall k \in \mathbb{N}^* : \lambda_k^2 = -\mu,$$

there exists a unique solution of $y \in V$ (orthogonal to the φ_k, for all $k \in \mathbb{N}^* : \lambda_k^2 = -\mu$) of (2.17).

Now we set

$$U_n = (y_n, \lambda y_n, \delta_n),$$

with

$$\delta_n = \eta_n \delta,$$

where $(\eta_n)_n$ is the sequence associated with x_0 built in Lemma 2.3.3 below and $y_n \in V$ is the solution of

$$\int_\Omega (\lambda^2 y_n \bar{w} + \nabla y_n \cdot \nabla \bar{w})\, dx = F_n(w), \forall w \in V, \tag{2.18}$$

where $F_n \in V'$ is defined by

$$F_n(w) := \sum_{k \in \mathbb{N}^* : \lambda_k^2 = -\lambda^2} \alpha_k^{(n)} \int_\Omega \varphi_k \bar{w}\, dx + \int_{\Gamma_0} (\delta_n, C)\bar{w}\, d\sigma, \forall w \in V,$$

with

$$\alpha_k^{(n)} = -\int_{\Gamma_0} (\delta_n, C)\bar{\varphi}_k\, d\sigma, \forall k \in \mathbb{N}^* : \lambda_k^2 = -\lambda^2,$$

for $\lambda \in \Sigma$ such that $\lambda^2 \in \sigma_d(\Delta_m)$.
If $\lambda^2 \notin \sigma_d(\Delta_m)$, we simply define F_n by

$$F_n(w) := \int_{\Gamma_0} (\delta_n, C)\bar{w}\, d\sigma, \forall w \in V.$$

The existence of a unique solution y_n of (2.18) in both cases is a consequence of the Fredholm alternative mentioned above.

Let us proceed first with the case $\lambda^2 \in \sigma_d(\Delta_m)$. The existence of y_n is justified by the fact that

$$F_n(\varphi_k) = 0, \forall k \in \mathbb{N}^* : \lambda_k^2 = -\lambda^2.$$

Before going further we notice that

$$\alpha_k^{(n)} \to 0 \text{ as } n \to \infty, \forall k \in \mathbb{N}^* : \lambda_k^2 = -\lambda^2. \tag{2.19}$$

Indeed, by definition we have

$$|\alpha_k^{(n)}| \lesssim \|\eta_n\|_{H^{-1/2}(\Gamma_0)} |\varphi_k|_{H^1(\Omega)} \lesssim \|\eta_n\|_{H^{-1/2}(\Gamma_0)} |\lambda_k|,$$

and by Lemma 2.3.3 below we obtain (2.19). A direct consequence of this property is that

$$\|F_n\|_{V'} \to 0 \text{ as } n \to \infty,$$

and again by the Fredholm alternative

$$\|y_n\|_V \to 0 \text{ as } n \to \infty. \tag{2.20}$$

Now applying Green's formula (as in Proposition 1.2.1), we see that $y_n \in V$ solution of (2.18) satisfies

$$\lambda^2 y_n - \Delta y_n = g_n := \sum_{k \in \mathbb{N}^* : \lambda_k^2 = -\lambda^2} \alpha_k^{(n)} \varphi_k \text{ in } \mathcal{D}'(\Omega). \tag{2.21}$$

as well as

$$\frac{\partial y_n}{\partial \nu} = (\delta_n, C), \text{ on } \Gamma_0. \tag{2.22}$$

Consequently, U_n belongs to $D(\mathcal{A})$ and

$$\lambda U_n - \mathcal{A} U_n = (0, g_n, (B(x_0) - B)\delta_n + \lambda C y_n). \tag{2.23}$$

Let us show that

$$\lambda U_n - \mathcal{A} U_n \to 0 \text{ in } \mathcal{H} \text{ as } n \to \infty. \tag{2.24}$$

Indeed, by definition we can write

$$\|g_n\|^2_{L^2(\Omega)} = \sum_{k \in \mathbb{N}^* : \lambda_k^2 = -\lambda^2} |\alpha_k^{(n)}|^2$$

and we directly deduce from (2.19) that

$$\|g_n\|_{L^2(\Omega)} \to 0 \text{ as } n \to \infty.$$

For the third component by the triangular inequality and a trace theorem, we have

$$\|(B(x_0) - B)\delta_n + \lambda C y_n\|_{L^2(\Gamma_0)} \lesssim \|(B(x_0) - B)\delta_n\|_{L^2(\Gamma_0)} + \|y_n\|_V.$$

Since the second term of this right-hand side tends to zero as n goes to infinity, it remains to estimate the first term: But B being uniformly continuous, we have

$$\forall \varepsilon > 0, \exists \eta_\varepsilon > 0 : |x - x_0| < \eta_\varepsilon \Rightarrow \|B(x_0) - B(x)\| < \varepsilon.$$

But by construction, the support of δ_n is included in $B(x_0, \varepsilon_n) \cap \Gamma_0$ with $\varepsilon_n \leq \frac{C}{n}$, for some $C > 0$ (independent of n). Hence, for a fixed $\varepsilon > 0$, and for $n > \frac{C}{\eta_\varepsilon}$, we have

$$\|B(x_0) - B(x)\| < \varepsilon, \forall x \in \text{ supp } \delta_n,$$

and we deduce that

$$\|(B(x_0) - B)\delta_n\|_{L^2(\Gamma_0)} \leq \varepsilon \|\delta_n\|_{L^2(\Gamma_0)} = \varepsilon \|\delta\|.$$

This shows that

$$\|(B(x_0) - B)\delta_n\|_{L^2(\Gamma_0)} \to 0 \text{ as } n \to \infty,$$

and finishes the proof of (2.24).

It remains to check that

$$\|U_n\|_{\mathcal{H}} \sim 1. \tag{2.25}$$

Indeed, by definition, we have

$$\|U_n\|^2_{\mathcal{H}} = \|y_n\|^2_V + |\lambda|^2 \|y_n\|^2_{L^2(\Omega)} + \|\delta_n\|^2_{L^2(\Gamma_0)},$$

hence by (2.20), we get

$$\|U_n\|^2_{\mathcal{H}} \sim \|\delta_n\|^2_{L^2(\Gamma_0)} = \|\delta\|^2 > 0.$$

If $\lambda^2 \notin \sigma_d(\Delta_m)$, then the solution $y_n \in V$ of (2.18) satisfies

$$\lambda^2 y_n - \Delta y_n = 0, \text{ in } \mathcal{D}'(\Omega),$$

and

$$\frac{\partial y_n}{\partial \nu} = (\delta_n, C), \text{ on } \Gamma_0,$$

thus U_n belongs to $D(\mathcal{A})$ and satisfies (2.25) with $g_n = 0$, i.e.

$$(\lambda U_n - \mathcal{A}U_n) = (0, 0, (B(x_0) - B)\delta_n + \lambda C y_n).$$

Moreover, it clearly satisfies both (2.24) and (2.25).

Note that (2.25) also implies that (U_n) has no convergent subsequence. Indeed, if a subsequence (U_{n_k}) is such that

$$U_{n_k} \to U \text{ in } \mathcal{H}, \text{ as } k \to \infty,$$

then by (2.20), $(\eta_{n_k})_k$ converges in $L^2(\Gamma_0)$, which contradicts Lemmas 2.3.3 and 2.3.4.

In conclusion, we have shown that $\lambda - \mathcal{A}$ is singular.

For any $\lambda \in \mathbb{C} \setminus \Sigma$, we denote by A_λ the linear (and continuous) operator from V into V' defined by

$$\langle A_\lambda u, v\rangle_{V'-V} := a_\lambda(u, v), \forall u, v \in V,$$

where a_λ is defined by (2.14) (well defined because $\lambda \in \mathbb{C} \setminus \Sigma$). According to the proof of Proposition 1.2.1, A_λ is an isomorphism for all positive real numbers. Hence, if we show that for any $\lambda, \mu \in \mathbb{C} \setminus \Sigma$, $A_\lambda - A_\mu$ is a compact operator, then by a standard perturbation result, A_λ will be a Fredholm operator of index zero for any $\lambda \in \mathbb{C} \setminus \Sigma$. To prove our compactness property, we notice that

$$\begin{aligned}\langle (A_\lambda - A_\mu)u, v\rangle_{V'-V} &= (\lambda^2 - \mu^2)\int_\Omega u\bar{v}dx \\ &+ \int_{\Gamma_0} (\lambda((\lambda I - B)^{-1}C, C) - \mu((\mu I - B)^{-1}C, C)\gamma_0 u \gamma_0 \bar{v} ds.\end{aligned}$$

Hence, due to the continuity of B and C, and Cauchy-Schwarz's inequality, we see that

$$\begin{aligned}\left|\langle (A_\lambda - A_\mu)u, v\rangle_{V'-V}\right| &\leq |\lambda^2 - \mu^2| \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &+ C(\lambda, \mu)\|u\|_{L^2(\Gamma_0)}\|v\|_{L^2(\Gamma_0)},\end{aligned}$$

where $C(\lambda, \mu)$ is a positive constant depending on λ and μ. Hence, by a trace theorem, we deduce that for any $\varepsilon \in (0, \frac{1}{2})$

$$\begin{aligned}\left|\langle (A_\lambda - A_\mu)u, v\rangle_{V'-V}\right| &\leq |\lambda^2 - \mu^2| \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &+ C(\lambda, \mu)C_\varepsilon\|u\|_V\|v\|_{H^{\frac{1}{2}+\varepsilon}(\Omega)},\end{aligned}$$

where C_ε is a positive constant depending on ε.

In conclusion, for any $\varepsilon \in (0, \frac{1}{2})$ if we set

$$H^{\frac{1}{2}+\varepsilon}_{\Gamma_1}(\Omega) = \{v \in H^{\frac{1}{2}+\varepsilon}(\Omega) : v = 0 \text{ on } \Gamma_1\},$$

that is clearly a Hilbert space equipped with the inner product of $H^{\frac{1}{2}+\varepsilon}(\Omega)$, we have shown that there exists a positive constant $C(\lambda, \mu, \varepsilon)$ depending on λ, μ and ε such that

$$\left|\langle (A_\lambda - A_\mu)u, v\rangle_{V'-V}\right| \leq C(\lambda, \mu, \varepsilon)\|u\|_V\|v\|_{H^{\frac{1}{2}+\varepsilon}(\Omega)}, \forall u, v \in V.$$

Equivalently, this means that

$$\sup_{v\in V, v\neq 0} \frac{\left|\langle (A_\lambda - A_\mu)u, v\rangle_{V'-V}\right|}{\|v\|_{H^{\frac{1}{2}+\varepsilon}(\Omega)}} \leq C(\lambda,\mu,\varepsilon)\|u\|_V.$$

Accordingly, as V is dense in $H^{\frac{1}{2}+\varepsilon}_{\Gamma_1}(\Omega)$, we deduce that $(A_\lambda - A_\mu)u$ belongs to $(H^{\frac{1}{2}+\varepsilon}_{\Gamma_1}(\Omega))'$ with

$$\begin{aligned}
\|(A_\lambda - A_\mu)u\|_{(H^{\frac{1}{2}+\varepsilon}_{\Gamma_1}(\Omega))'} &= \sup_{v\in H^{\frac{1}{2}+\varepsilon}_{\Gamma_1}(\Omega), v\neq 0} \frac{\left|\langle (A_\lambda - A_\mu)u, v\rangle_{V'-V}\right|}{\|v\|_{H^{\frac{1}{2}+\varepsilon}(\Omega)}}\\
&= \sup_{v\in V, v\neq 0} \frac{\left|\langle (A_\lambda - A_\mu)u, v\rangle_{V'-V}\right|}{\|v\|_{H^{\frac{1}{2}+\varepsilon}(\Omega)}}\\
&\leq C(\lambda,\mu,\varepsilon)\|u\|_V.
\end{aligned}$$

As V is compactly and densely embedded into $H^{\frac{1}{2}+\varepsilon}_{\Gamma_1}(\Omega)$, by duality, $(H^{\frac{1}{2}+\varepsilon}_{\Gamma_1}(\Omega))'$ is also compactly embedded into V' and therefore $A_\lambda - A_\mu$ is a compact operator from V into V'.

Now we readily check that, for any $\lambda \in \mathbb{C}\setminus\Sigma$, we have the equivalence

$$y \in \ker A_\lambda \iff (y, \lambda y, -\lambda(\lambda I - B)^{-1}C\gamma_0 y)^\top \in \ker(\lambda I - \mathcal{A}). \tag{2.26}$$

This equivalence implies that for any $\lambda \in \mathbb{C}\setminus\Sigma$, $\ker(\lambda I - \mathcal{A})$ is always finite-dimensional and has the same dimension as $\ker A_\lambda$. This last property follows from the fact (used below) that the expression

$$(y,z)_{\lambda,V} := \Big((y, \lambda y, -\lambda(\lambda I - B)^{-1}C\gamma_0 y)^\top, (z, \lambda z, -\lambda(\lambda I - B)^{-1}C\gamma_0 z)^\top\Big)_{\mathcal{H}},$$

is an inner product on V whose associated norm is equivalent to the standard one. Denote by $\{y^{(i)}\}_{i=1}^N$ an orthonormal basis of $\ker A_\lambda$ for this new inner product (for shortness the dependence of λ is dropped), i.e.

$$(y^{(i)}, y^{(j)})_{\lambda,V} = \delta_{ij}, \forall i,j = 1,\dots,N.$$

Finally, for all $i = 1,\dots,N$, we set

$$Z^{(i)} = (y^{(i)}, \lambda y^{(i)}, -\lambda(\lambda I - B)^{-1}C\gamma_0 y^{(i)})^\top,$$

the element of $\ker(\lambda I - \mathcal{A})$ associated with $y^{(i)}$ that are orthonormal with respect to the inner product of $\mathcal{H}$.

Let us now show that for all $\lambda \in \mathbb{C}\setminus\Sigma$, the range $R(\lambda I - \mathcal{A})$ of $\lambda I - \mathcal{A}$ is closed. Indeed, let us consider a sequence $U_n = (y_n, z_n, \delta_n)^\top \in D(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A})U_n = F_n = (y_{1n}, z_{1n}, \delta_{1n})^\top \to F = (y_1, z_1, \delta_1)^\top \text{ in } \mathcal{H}. \tag{2.27}$$

Without loss of generality we can assume that

$$(U_n, Z^{(i)})_{\mathcal{H}} = -\alpha_{n,i}, \forall i = 1,\dots,N. \tag{2.28}$$

where

$$\alpha_{n,i} := ((0, y_{1n}, -(\lambda I - B)^{-1}(\delta_{1n} + C\gamma_0 y_{1n}))^\top, Z^{(i)})_{\mathcal{H}}.$$

Indeed, if this is not the case, we can consider

$$\tilde{U}_n = U_n - \sum_{i=1}^{N} \beta_i Z^{(i)}$$

that still belongs to $D(\mathcal{A})$ and satisfies

$$(\lambda I - \mathcal{A})\tilde{U}_n = F_n,$$

as well as

$$(\tilde{U}_n, Z^{(i)})_{\mathcal{H}} = -\alpha_{n,i}, \forall i = 1, \ldots, N,$$

by setting

$$\beta_i = (U_n, Z^{(i)})_{\mathcal{H}} + \alpha_{n,i}, \forall i = 1, \ldots, N.$$

Note that the condition (2.28) is equivalent to

$$(y_n, y^{(i)})_{\lambda,V} = 0, \forall i = 1, \ldots, N.$$

In other words,

$$y_n \in (\ker A_\lambda)^{\perp_{\lambda,V}}, \tag{2.29}$$

where $^{\perp_{\lambda,V}}$ means that the orthogonality is taken with respect to the inner product $(\cdot,\cdot)_{\lambda,V}$.

Returning to (2.27), the arguments of the proof of Proposition 1.2.1 imply that

$$A_\lambda y_n = L_{F_n} \text{ in } V',$$

where L_F was defined by (2.15). But it is easy to check that

$$L_{F_n} \to L_F \text{ in } V'.$$

Moreover, as $\lambda \in \mathbb{C} \setminus \Sigma$, A_λ is an isomorphism from $(\ker A_\lambda)^{\perp_{\lambda,V}}$ into $R(A_\lambda)$, hence by (2.29) we deduce that there exists a positive constant $C(\lambda)$ such that

$$\|y_n - y_m\|_V \le C(\lambda)\|L_{F_n} - L_{F_m}\|_{V'}, \forall n, m \in \mathbb{N}.$$

Hence, $(y_n)_n$ is a Cauchy sequence in V, and therefore there exists $y \in V$ such that

$$y_n \to y \text{ in } V,$$

as well as

$$A_\lambda y = L_F \text{ in } V'.$$

Setting $z = \lambda y - y_1$ and $\delta = (\lambda I - B)^{-1}(\delta_1 - C\gamma_0 z)$, we deduce that $U := (y, z, \delta)^\top$ belongs to $D(\mathcal{A})$ and

$$(\lambda I - \mathcal{A})U = F.$$

In other words, F belongs to $R(\lambda I - \mathcal{A})$. The closedness of $R(\lambda I - \mathcal{A})$ is thus proved.

At this stage, for any $\lambda \in \mathbb{C} \setminus \Sigma$, we show that

$$\text{codim } R(A_\lambda) = \text{ codim } R(\lambda I - \mathcal{A}), \tag{2.30}$$

where codim W is the dimension of the orthogonal of W.

Indeed, let us set $N =$ codim $R(A_\lambda)$, then there exist N elements $\varphi_i \in V, i = 1, \ldots, N$ such that

$$f \in R(A_\lambda) \iff f \in V' \text{ and } \langle f, \varphi_i \rangle_{V'-V} = 0, \forall i = 1, \ldots, N.$$

Consequently, for $F \in \mathcal{H}$, if L_F (that belongs to V') satisfies

$$L_F(\varphi_i) = 0, \forall i = 1, \ldots, N, \tag{2.31}$$

there exists a solution $y \in V$ of

$$A_\lambda y = L_F \text{ in V'},$$

and as usual the arguments of the proof of Proposition 1.2.1 implies that F is in $R(\lambda I - \mathcal{A})$. Hence, the N conditions on $F \in \mathcal{H}$ from (2.31) allow to show that it belongs to $R(\lambda I - \mathcal{A})$, and therefore

$$\text{codim } R(\lambda I - \mathcal{A}) \leq N = \text{ codim } R(A_\lambda). \tag{2.32}$$

This shows that $\lambda I - \mathcal{A}$ is a Fredholm operator.

Conversely, set $M =$ codim $R(\lambda I - \mathcal{A})$, then there exist M elements $\Psi_i = (y_i, z_i, \delta_i) \in \mathcal{H}, i = 1, \ldots, M$ such that

$$F \in R(\lambda I - \mathcal{A}) \iff F \in \mathcal{H} \text{ and } (F, \Psi_i)_\mathcal{H} = 0, \forall i = 1, \ldots, M.$$

Then, for any $f \in L^2(\Omega)$, if

$$(f, z_i)_{L^2(\Omega)} = ((0, f, 0)^\top, \Psi_i)_\mathcal{H} = 0, \forall i = 1, \ldots, M, \tag{2.33}$$

there exists $U = (y, z, \delta)^\top \in D(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A})U = (0, f, 0),$$

or equivalently (using the definition of $\mathcal{A}$ and the invertibility of $\lambda I - B$)

$$\begin{aligned} z &= \lambda y, \\ \lambda^2 y - \Delta y &= f, \\ \delta &= -\lambda(\lambda I - B)^{-1} C \gamma_0 y. \end{aligned}$$

Multiplying this second identity by $\varphi \in V$, integrating in Ω and using Green's formula (2.6), we obtain that

$$a_\lambda(y, \varphi) = \int_\Omega f \bar{\varphi} dx, \forall \varphi \in V.$$

This shows that

$$R(A_\lambda) \supset \{f \in L^2(\Omega) \text{ satisfying (2.33)}\}.$$

Hence,

$$\text{codim } R(A_\lambda) \leq M = \text{ codim } R(\lambda I - \mathcal{A}). \tag{2.34}$$

The inequalities (2.32) and (2.34) imply (2.30).

We conclude the second point by using the fact that A_λ is a Fredholm operator of index zero for any $\lambda \in \mathbb{C} \setminus \Sigma$, the equivalence (2.26) and the identity (2.30). ■

Corollary 2.3.2 *$\lambda \notin \Sigma$ if and only if $\lambda - \mathcal{A}$ is a Fredholm operator of index zero.*

Lemma 2.3.3 *For every interior point x_0 of Γ_0, there exists a sequence $(\eta_n)_{n\in\mathbb{N}^*}$ of functions in $L^2(\Gamma_0)$ such that*

$$\|\eta_n\|_{L^2(\Gamma_0)} = 1, \forall n \in \mathbb{N}^*, \tag{2.35}$$

as well as

$$\|\eta_n\|_{H^{-\frac{1}{2}}(\Gamma_0)} \to 0 \text{ as } n \to \infty. \tag{2.36}$$

Moreover, the support of η_n is included in $B(x_0, \varepsilon_n) \cap \Gamma_0$ with $\varepsilon_n \sim \frac{1}{n}$ and therefore the sequence $(\eta_n)_{n\in\mathbb{N}^*}$ has no convergent subsequence in $L^2(\Gamma_0)$.

Proof. By definition of the regularity of the boundary (see for instance [25, Def 1.2.1.1]), there exist a neighborhood W of x_0 in $\mathbb{R}^d$ and a local system of cartesian coordinates (y', y_d) and a Lipschitz mapping φ from W' the projection of W on $\mathbb{R}^{d-1}$ to $\mathbb{R}$ such that W is a hypercube and

$$\begin{aligned} \Omega \cap W &= \{(y', y_d) \in W : y_d < \varphi(y')\}, \\ \Gamma_0 \cap W &= \{(y', y_d) \in W : y_d = \varphi(y')\}. \end{aligned}$$

Denote by y'_0 the point in W' such that

$$(y'_0, \varphi(y'_0)) = x_0.$$

Fix a function $\eta \in \mathcal{D}(\mathbb{R}^{d-1})$ with a support in $B(0,1)$ and such that

$$\|\eta\|_{L^2(\mathbb{R}^{d-1})} = 1.$$

Then for n large enough namely such that $\bar{B}(y'_0, \frac{1}{n}) \subset W'$, we take

$$\eta_n(y', \varphi(y')) = n^{\frac{d-1}{2}} \eta(n(y' - y'_0)), \forall y' \in W'$$

and extended by zero outside $\Gamma_0 \cap W$.

We directly check that the support of η_n is (in this proof $\|\cdot\|_2$ means the Euclidean norm of $\mathbb{R}^{d-1}$ or $\mathbb{R}^d$)

$$S_n = \{(y', \varphi(y')) : \|y' - y'_0\|_2 \leq \frac{1}{n}\}.$$

Hence, for $(y', \varphi(y')) \in S_n$, we have

$$\|(y', \varphi(y')) - (y'_0, \varphi(y'_0))\|_2 \sim \|y' - y'_0\|_2 + |\varphi(y') - \varphi(y'_0)| \sim \|y' - y'_0\|_2,$$

and the property on the support of η_n follows.

Now by a change of variables we see that

$$\|\eta_n\|^2_{L^2(\Gamma_0)} \sim n^{d-1} \int_{W'} |\eta(n(y' - y'_0))|^2 \, dy' = \int_{\mathbb{R}^{d-1}} |\eta(z)|^2 \, dz,$$

and the property (2.35) holds (up to a multiplicative factor equivalent to 1).

To prove (2.36), as $L^2(\Gamma_0)$ is compactly embedded into $H^{-\frac{1}{2}}(\Gamma_0)$, by (2.35) there exists a subsequence, still denoted by (η_n), such that

$$\eta_n \to \eta \text{ in } H^{-\frac{1}{2}}(\Gamma_0) \text{ as } n \to \infty,$$

for some $\eta \in H^{-\frac{1}{2}}(\Gamma_0)$. But this property implies that

$$\eta_n \to \eta \text{ in } \mathcal{D}'(\Gamma_0) \text{ as } n \to \infty.$$

As we will show that

$$\eta_n \to 0 \text{ in } \mathcal{D}'(\Gamma_0) \text{ as } n \to \infty, \tag{2.37}$$

we deduce that $\eta = 0$ and (2.36) follows.

In the same manner, if (η_n) would have a convergent subsequence in $L^2(\Gamma_0)$, then by (2.37), this subsequence would converge to 0 in $L^2(\Gamma_0)$, which contradicts (2.35).

It then remains to show (2.37). For that purpose, fix $\psi \in \mathcal{D}(\Gamma_0)$, then by a change of variables, we may write

$$\begin{aligned}
\langle \eta_n, \psi \rangle &= \int_{\Gamma_0} \eta_n(x)\psi(x)\, d\sigma \\
&= n^{\frac{d-1}{2}} \int_{W'} \eta(n(y'-y_0'))\psi(y',\varphi(y'))\sqrt{1+\sum_{i=1}^{d-1}|\frac{\partial\varphi}{\partial y_i}(y')|^2}\, dy' \\
&= n^{\frac{1-d}{2}}\gamma_n(\psi),
\end{aligned}$$

where

$$\gamma_n(\psi) = \int_{B(0,1)} \eta(z)\psi(y_0'+\frac{z}{n},\varphi(y_0'+\frac{z}{n}))\sqrt{1+\sum_{i=1}^{d-1}|\frac{\partial\varphi}{\partial y_i}(y_0'+\frac{z}{n})|^2}\, dz.$$

Since φ is Lipschitz, we deduce that

$$|\gamma_n(\psi)| \lesssim \int_{B(0,1)} |\eta(z)| \left|\psi(y_0'+\frac{z}{n},\varphi(y_0'+\frac{z}{n}))\right| dz.$$

As

$$\int_{B(0,1)} |\eta(z)| \left|\psi(y_0'+\frac{z}{n},\varphi(y_0'+\frac{z}{n}))\right| dz \to |\psi(x_0)| \int_{B(0,1)} |\eta(z)|\, dz \text{ as } n \to \infty,$$

we have shown that $\gamma_n(\psi)$ remains bounded as n becomes large. Therefore, we deduce that

$$\langle \eta_n, \psi \rangle \to 0 \text{ as } n \to \infty,$$

which proves (2.37). ■

Lemma 2.3.4 *Let $x_0 \in \partial\Gamma_0$. Then the statements of Lemma 2.3.3 remain true.*

Proof. As in the previous lemma there exist a neighborhood W of x_0 in $\mathbb{R}^d$, a local system of cartesian coordinates (y', y_d) and a Lipschitz mapping φ from W' the projection of W on $\mathbb{R}^{d-1}$ to $\mathbb{R}$ such that W is a hypercube and

$$\begin{aligned}
\Omega \cap W &= \{(y', y_d) \in W : y_d < \varphi(y')\}, \\
\Gamma \cap W &= \{(y', y_d) \in W : y_d = \varphi(y')\}.
\end{aligned}$$

Denote by y_0' the point in W' such that

$$(y_0', \varphi(y_0')) = x_0,$$

and set

$$\Phi(y') = (y', \varphi(y')), \forall y' \in W',$$

that is a bijection from W' into $\Gamma \cap W$. Without loss of generality we may suppose that $y_0' = 0$. Denote by Γ_0' the set $\Phi^{-1}(\Gamma_0 \cap W) \subset W'$, and by $I' \subset W'$ the set $I' = \Phi^{-1}(\Gamma_0 \cap \Gamma_1 \cap W)$. Our assumption that $\Gamma_0 \cap \bar{\Gamma}_1$ is C^1 means that the curve I' is a C^1 curve in W', in other words, there exist a local system of Cartesian coordinates (z'', z_{d-1}) and a C^1 mapping ψ such that I' coincides near 0 with the curve

$$\{(z'', \psi(z'')) : z'' \in W''\},$$

while Γ_0' coincides near 0 with

$$\{(z'', z_{d-1}) : z_{d-1} > \psi(z''), \forall z'' \in W''\},$$

where again W'' is a hypercube of $\mathbb{R}^{d-2}$. Again without loss of generality we can assume that $(0, \psi(0)) = 0$ as well as $\nabla\psi(0) = 0$.

Now instead of using the coordinates y', we use the coordinates (z'', z_{d-1}) (and replace W' by another hypercube $\widetilde{W''}$) and as before we perform the change of variables

$$\begin{cases} \hat{z}_{d-1} = nz_{d-1}, \\ \hat{z}'' = nz''. \end{cases}$$

The difficulty lies in the fact that the curve I' becomes now the curve

$$\hat{z}_{d-1} = n\psi(\frac{\hat{z}''}{n}),$$

that is tangent to the hyperplane $\hat{z}_{d-1} = 0$ but depends on n and similarly the domain Γ_0' becomes the domain

$$\hat{z}_{d-1} > n\psi(\frac{\hat{z}''}{n}),$$

that also depends on n.

But the regularity on ψ allows to show that there exists $\varepsilon > 0$ small enough and n_0 large enough such that for all $n > n_0$

$$n|\psi(\frac{\hat{z}''}{n})| \leq \varepsilon, \forall \|\hat{z}''\|_2 \leq 1. \tag{2.38}$$

Indeed, for $\hat{z}''$ fixed such that $\|\hat{z}''\| \leq 1$ and by considering the mapping $f(t) = \psi(\frac{t\hat{x}''}{n})$, we can write

$$f(1) = \int_0^1 \nabla\psi\left(\frac{t\hat{x}''}{n}\right) \cdot \frac{\hat{x}''}{n} dt,$$

and deduce,

$$n\left|\psi\left(\frac{\hat{z}''}{n}\right)\right| \leq \sup_{\|\hat{w}''\|_2 \leq 1} \left|\nabla\psi\left(\frac{\hat{w}''}{n}\right)\right|.$$

This leads to (2.38) because

$$\sup_{\|\hat{w}''\|_2 \leq 1} \left| \nabla \psi \left(\frac{\hat{w}''}{n} \right) \right| = \sup_{\|w''\|_2 \leq \frac{1}{n}} \left| \nabla \psi \left(w'' \right) \right|,$$

that tends to zero as n goes to infinity.

Now we mainly proceed as before: we fix a function $\eta \in \mathcal{D}(\mathbb{R}^{d-1})$ with a support included in $\{(\hat{x}'', \hat{x}_{d-1}) \in B(0,1) : \hat{x}_{d-1} > 2\varepsilon\}$ and such that

$$\|\eta\|_{L^2(\mathbb{R}_+^{d-1})} = 1.$$

Then for n large enough, we take

$$\eta_n((z'', z_{d-1}), \varphi(z'', z_{d-1})) = n^{\frac{d-1}{2}} \eta((nz'', nz_{d-1}))), \forall (z'', z_{d-1}) \in \widetilde{W}''$$

and extended by zero outside its support.

We directly check that the support of η_n is of size of order $\frac{1}{n}$ and is included in $\Gamma_0 \cap W$. At this stage the proof is continued as in the previous Lemma. ■

2.4 Strong stability

In [53] and [2], where the problem is one dimensional in space (i.e. $d = 1$), the strong stability was proven using Arendt-Batty theorem (see [10] and Theorem 1.3.1) since the resolvent of the infinitesimal generator considered therein is compact and therefore the study of $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is reduced to the study of purely complex eigenvalues of $\mathcal{A}$. In our case, as $D(\mathcal{A})$ is not compactly embedded in $\mathcal{H}$, this method partially fails to achieve the proof of strong stability. Nevertheless, with similar assumptions on B as those of the one-dimensional case, we are able to obtain the strong stability by using Arendt-Batty theorem (see Theorem 1.3.1).

In view of Theorem 2.3.1, $\sigma(\mathcal{A})$ is not purely formed of eigenvalues and therefore we have to analyze $\sigma_d(\mathcal{A}) \cap i\mathbb{R}$ as well as $\sigma(\mathcal{A}) \setminus \sigma_d(\mathcal{A})) \cap i\mathbb{R}$.

We start with the eigenvalues of $\mathcal{A}$ on the imaginary axis.

Proposition 2.4.1 *$\mathcal{A}$ is an isomorphism if and only if* $0 \notin \Sigma$.

Proof. By Corollary 2.3.2, $0 \notin \Sigma$ if and only if $\mathcal{A}$ is a Fredholm operator of index zero. Hence, the conclusion follows if we show that

$$\ker \mathcal{A} = \{0\},$$

or equivalently, due to the proof of Theorem 2.3.1,

$$\ker A_0 = \{0\}.$$

But in view of the definition of A_0, $y \in \ker A_0$ if and only if $y \in V$ is solution of

$$a_0(y, \varphi) = \int_\Omega \nabla y \nabla \bar{\varphi} dx = 0, \forall \varphi \in V.$$

From the coerciveness of a_0 on V, we conclude that $y = 0$. ■

For the other eigenvalues on the imaginary axis, we need the positive self-adjoint operator $-\Delta_{Dir}$ defined by

$$D(-\Delta_{Dir}) = \{y \in H_0^1(\Omega) : \Delta y \in L^2(\Omega)\}$$

and

$$-\Delta_{Dir} y = -\Delta y, \forall y \in D(-\Delta_{Dir}).$$

Denote by $\sigma(-\Delta_{Dir}) := \{\lambda_{Dir,k}^2\}_{k\in\mathbb{N}^*}$ the (discrete) spectrum of $-\Delta_{Dir}$ (repeated according to their multiplicity) and let $y_{Dir,k}$ be the corresponding orthonormalized eigenvectors.

Proposition 2.4.2 *Assume that*

(A_1) $\forall iz \notin \Sigma, z \in \mathbb{R}^*, \exists \alpha_z > 0 : \Re((izI - B(x))^{-1}C(x), C(x)) \geq \alpha_z, \forall x \in \Gamma_0,$

(A_2) $\Sigma \cap \{\pm i\lambda_{Dir,k}, k \in \mathbb{N}^*\} = \emptyset,$

(A_3) $\forall iz \in \Sigma : C \notin \ker(izI + B^*)^\perp$ *on* $\Gamma_0,$

(A_4) $\forall iz \in \Sigma : \forall M \subset \Gamma_0 :$ *meas* $M > 0 : \exists x \in M : (\eta, C(x))_x \neq 0$ *for all nonzero* $\eta \in \ker(izI - B(x)).$

Then

$$\sigma_d(\mathcal{A}) \cap i\mathbb{R}^* = \emptyset. \tag{2.39}$$

Proof. Assume that $i\lambda$ is a non zero eigenvalue of $\mathcal{A}$ in $i\mathbb{R}$. Let $U = (y, z, \delta)^\top \in D(\mathcal{A})$, $U \neq 0$ be the associated eigenvector. Then, we have

$$\mathcal{A}U = i\lambda U,$$

which implies $z = i\lambda y$,

$$-\lambda^2 y - \Delta y = 0 \text{ in } \Omega,$$

as well as

$$(i\lambda - B)\delta = -i\lambda C y \text{ on } \Gamma_0. \tag{2.40}$$

Now we distinguish two cases:

i) if $i\lambda \notin \Sigma$, then by the proof of Proposition 1.2.1, we deduce that $y \in V$ satisfies

$$a_{i\lambda}(y, \varphi) = 0, \forall \varphi \in V.$$

In particular, taking $\varphi = y$, we get

$$a_{i\lambda}(y, y) = 0.$$

Taking the imaginary part of this identity, we find

$$-\lambda \int_{\Gamma_0} \Re((i\lambda I - B(x))^{-1}C(x), C(x))|\gamma_0 y(x)|^2 \, d\sigma = 0.$$

Since λ is different from zero, we get

$$\int_{\Gamma_0} \Re((i\lambda I - B(x))^{-1}C(x), C(x))|\gamma_0 y(x)|^2 \, d\sigma = 0,$$

and by the assumption (A_1) we find that

$$y = 0 \text{ on } \Gamma_0.$$

Due to (2.40) and since $i\lambda \notin \Sigma$, we deduce that $\delta = 0$ and then $\frac{\partial y}{\partial \nu} = (\delta, C) = 0$, thus y satisfies

$$\begin{cases} \lambda^2 y - \Delta y = 0 & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma, \\ \frac{\partial y}{\partial \nu} = 0 & \text{on } \Gamma_0, \end{cases} \tag{2.41}$$

By Holmgren's theorem we deduce that $y = 0$, which is impossible (otherwise U would be zero).
ii) if $i\lambda \in \Sigma$, then we again distinguish two cases:
a) if $y = 0$ (then $\delta \neq 0$ on a set M of positive measure) and by (2.40), we find

$$(i\lambda - B)\delta = 0 \text{ on } \Gamma_0.$$

On the other hand, (2.5) here implies

$$(\delta, C) = 0 \text{ on } \Gamma_0,$$

which is in contradiction with (A_4).
b) if $y \neq 0$, then (2.40) implies that

$$Cy \in R(i\lambda - B) = \ker(i\lambda I + B^*)^{\perp} \text{ on } \Gamma_0.$$

Hence, by our assumption (A_3), we find that

$$y = 0 \text{ on } \Gamma_0.$$

This implies that $y \in H_0^1(\Omega)$ satisfies

$$\int_\Omega (-\lambda^2 y\bar{\varphi} + \nabla y \cdot \nabla \bar{\varphi})\, dx = 0, \forall \varphi \in H_0^1(\Omega) \subset V.$$

Consequently, $y \in D(-\Delta_{Dir})$ and satisfies

$$-\Delta_{Dir} y = \lambda^2 y.$$

We have shown that there exists $k \in \mathbb{N}^*$ such that $\lambda^2 = \lambda^2_{Dir,k}$. Coming back to (2.40), we see that

$$(\pm i\lambda_{Dir,k} - B)\delta = 0 \text{ on } \Gamma_0.$$

From our assumption (A_2) we deduce that

$$\delta = 0 \text{ on } \Gamma_0.$$

This property and the boundary condition (2.5) then imply

$$\frac{\partial y}{\partial \nu} = 0 \text{ on } \Gamma_0.$$

By Holmgren's theorem we deduce that $y = 0$, which is impossible.
■

Proposition 2.4.3 *If (A_1) to (A_4) from the previous proposition hold, if $0 \notin \Sigma$ and if $\Sigma \cap i\mathbb{R}$ is countable, then the C_0-semigroup associated with $\mathcal{A}$ is strongly stable.*

Proof. Propositions 2.4.1 and 2.4.2 guarantee that

$$\sigma_d(\mathcal{A}) \cap i\mathbb{R} = \emptyset. \tag{2.42}$$

Let us now show that

$$\sigma(\mathcal{A}) \cap i\mathbb{R} \subset \Sigma \cap i\mathbb{R}. \tag{2.43}$$

Indeed, if $i\lambda \in \sigma(\mathcal{A}) \cap i\mathbb{R}$, then either $i\lambda$ is in Σ as required, or $i\lambda$ is not in Σ, but then by Theorem 2.3.1, $i\lambda$ belongs to $\sigma_d(\mathcal{A}) \cap i\mathbb{R}$, which is impossible due to (2.42).

The two properties (2.42), (2.43) and the assumption that $\Sigma \cap i\mathbb{R}$ is countable finally allow to apply the theorem of Arendt-Batty. ■

To end up this section, in the case when B and C are constant (in that case $\Sigma = \sigma(B)$), let us show that the sufficient conditions from Proposition 2.4.3 are “almost" necessary. Namely we prove the following result.

Proposition 2.4.4 *Assume that B and C are constant on Γ_0 and that (A_1) holds, which in this case reduces to*

$$\forall iz \notin \sigma(B), z \in \mathbb{R}^*, \Re((izI - B)^{-1}C, C) > 0.$$

Then (A_2), (A_3), (A_4) and $0 \notin \sigma(B)$ hold if and only if the C_0-semigroup associated with $\mathcal{A}$ is strongly stable.

Proof. As $\Sigma \cap i\mathbb{R} = \sigma(B) \cap i\mathbb{R}$ is finite, by the previous Proposition, the conditions (A_2), (A_3), (A_4) and $0 \notin \sigma(B)$ are clearly sufficient (since (A_1) holds). Hence, it suffices to show that they are also necessary. For that purpose, we show that if (A_2), (A_3), (A_4) or $0 \notin \sigma(B)$ does not hold, then $\mathcal{A}$ has an eigenvalue on the imaginary axis (since this condition directly implies that the C_0-semigroup associated with $\mathcal{A}$ is not strongly stable).

Firstly, if we assume that $0 \in \sigma(B)$, then there exists a nonzero $\delta \in \mathbb{C}^n$ such that

$$B\delta = 0.$$

Hence, we consider $y \in V$ solution of

$$\begin{cases} \Delta y = 0 \text{ in } \Omega, \\ \frac{\partial y}{\partial \nu} = (\delta, C) \text{ on } \Gamma_0. \end{cases}$$

Such a solution exists and is the unique solution $y \in V$ of

$$\int_\Omega \nabla y \cdot \nabla \bar{\varphi}\, dx = (\delta, C) \int_{\Gamma_0} \bar{\varphi}\, d\sigma, \forall \varphi \in V.$$

Since we easily check that $(y, 0, \delta)$ belongs to $\ker \mathcal{A}$, we deduce that 0 is an eigenvalue of $\mathcal{A}$.

Secondly assume that (A_2) does not hold, then this means that there exists $k \in \mathbb{N}^*$ such that

$$i\lambda_{Dir,k} \in \sigma(B) \text{ or } -i\lambda_{Dir,k} \in \sigma(B).$$

Assume that $i\lambda_{Dir,k} \in \sigma(B)$ (the other case is treated in the same manner), then there exists a non-zero $\delta_k \in \mathbb{C}^n$ such that

$$(i\lambda_{Dir,k} I - B)\delta_k = 0.$$

Now we distinguish the case $(\delta_k, C) = 0$ or not:
i) if $(\delta_k, C) = 0$, then we take $(0, 0, \delta_k)$ that belongs to $\ker(i\lambda_{Dir,k} I - \mathcal{A})$ as easily checked.

ii) if $(\delta_k, C) \neq 0$, then $(y_{Dir,k}, i\lambda_{Dir,k}y_{Dir,k}, \alpha_k\delta_k)$ belongs to $\ker(i\lambda_{Dir,k}I - \mathcal{A})$ with α_k being chosen as

$$\alpha_k = (\delta_k, C)^{-1}\frac{\partial y_{Dir,k}}{\partial \nu} \text{ on } \Gamma_0.$$

Thirdly, if (A_3) does not hold, then there exists $iz \in \Sigma$ such that

$$C \in \ker(izI + B^*)^{\perp} = R(izI - B). \tag{2.44}$$

This means that there exists $\delta_C \in \mathbb{C}^n$ such that

$$(izI - B)\delta_C = C. \tag{2.45}$$

Consider $y \in V$ the solution of (compare with (2.17))

$$\int_\Omega (-z^2 y\bar{w} + \nabla y \cdot \nabla \bar{w})\, dx = \int_{\Gamma_0} h\gamma_0\bar{w}\, d\sigma, \forall w \in V, \tag{2.46}$$

with $h \in L^2(\Gamma_0)$ arbitrary if $z^2 \notin \sigma_d(-\Delta_m)$. If $z^2 \in \sigma_d(-\Delta_m)$, then we fix h satisfying

$$\int_{\Gamma_0} h\gamma_0\bar{\varphi}_k\, d\sigma = 0, \forall k \in \mathbb{N}^* : \lambda_k^2 = z^2. \tag{2.47}$$

Hence, the Fredholm alternative (see the proof of Theorem 2.3.1) implies the existence of a solution y of (2.46).

Such a $h \in L^2(\Gamma_0)$ always exists. Indeed, assume that there exists $k \in \mathbb{N}^*$ such that $\lambda_k^2 = z^2$. Then we notice that the trace of $\gamma_0\varphi_k, k \in \mathbb{N}^*$ such that $\lambda_k^2 = z^2$ are linearly independent as element of $L^2(\Gamma_0)$, indeed if there exists $\alpha_k \in \mathbb{C}$ such that

$$\sum_{k\in\mathbb{N}^*:\lambda_k^2=z^2} \alpha_k\gamma_0\varphi_k = 0 \text{ in } L^2(\Gamma_0),$$

then

$$\varphi := \sum_{k\in\mathbb{N}^*:\lambda_k^2=z^2} \alpha_k\varphi_k,$$

is still an eigenvector of $-\Delta_m$ (of eigenvalue z^2) and satisfies the additional Dirichlet boundary condition:

$$\gamma_0\varphi = 0 \text{ in } L^2(\Gamma_0).$$

Hence, by Homlgren's theorem, $\varphi = 0$ in Ω and therefore $\alpha_k = 0$ for all $k \in \mathbb{N}^*$ such that $\lambda_k^2 = z^2$. Let $\{\psi_k : k \in \mathbb{N}^*, \lambda_k^2 = z^2\}$ in $L^2(\Gamma_0)$ be the orthonormal system constructed by the Gram-Schmidt process. Then starting with an arbitrary $h_0 \in L^2(\Gamma_0)$, the function h given by

$$h = h_0 - \sum_{k\in\mathbb{N}^*:\lambda_k^2=z^2} \left(\int_{\Gamma_0} h_0\gamma_0\bar{\psi}_k\, d\sigma\right)\psi_k,$$

fulfills (2.47).

Returning to (2.46), by Green's formula, we see that $y \in V$ satisfies

$$\begin{cases} -z^2y - \Delta y = 0 \text{ in } \Omega, \\ \frac{\partial y}{\partial \nu} = h \text{ on } \Gamma_0. \end{cases}$$

Now due to $iz \in \Sigma$ and (2.45), we see that

$$\delta := -izy\delta_C + \alpha\delta_0, \tag{2.48}$$

with $0 \neq \delta_0 \in \ker(izI - B)$ and any function $\alpha \in L^2(\Gamma_0)$ satisfies

$$(izI - B)\delta = -izCy \text{ on } \Gamma_0.$$

Again we distinguish the case $(C, \delta_0) = 0$ or not.
i) if $(C, \delta_0) = 0$, then we take $(0, 0, \delta_0)$ and check that it belongs to $\ker(izI - \mathcal{A})$.
ii) if $(C, \delta_0) \neq 0$, then we take $(y, izy, \delta)^\top$, where $y \in V$ is the unique solution of (2.46) (with a $h \in L^2(\Gamma_0)$ fulfilling (2.47)) and δ given by (2.48) with

$$\alpha := (\delta_0, C)^{-1}(h + izy(\delta_C, C)).$$

In that way, the triple $(y, izy, \delta)^\top$ fulfills

$$\frac{\partial y}{\partial \nu} = (\delta, C) \text{ on } \Gamma_0,$$

and hence belongs to $D(\mathcal{A})$. Again, easy calculations lead to $(izI - \mathcal{A})(y, izy, \delta)^\top = 0$.

Finally, if (A_4) does not hold, then there exists $iz \in \Sigma$ and a non zero $\delta \in \mathbb{C}^n$ such that

$$(izI - B)\delta = 0$$

with $(\delta, C) = 0$. In that case, we take $(0, 0, \delta)$ and easily check that it belongs to $\ker(izI - \mathcal{A})$. ■

2.5 Non uniform stability of $\mathcal{A}$

As before, in [53] and [2] since the problem is one dimensional in space, a perturbation result (see [44,46]) was used to prove the non uniform stability of the generated C_0-semigroup generated by $\mathcal{A}$. In our case this cannot be used to prove the non uniform stability. But adapting a method from [50] we can prove a non uniform stability result. This method is based on a frequency domain approach, namely we use the following result, called Huang-Prüss Theorem (see [23], [43] or [26]):

Lemma 2.5.1 *A C_0-semigroup $e^{t\mathcal{L}}$ of contractions on a Hilbert space H is exponentially stable, i.e., satisfies*

$$||e^{t\mathcal{L}}U_0||_H \leq C\, e^{-\omega t}||U_0||_H, \quad \forall U_0 \in H, \quad \forall t \geq 0,$$

for some positive constants C and ω if and only if

$$\rho(\mathcal{L}) \supset i\mathbb{R}, \tag{2.49}$$

and

$$\sup_{\beta \in \mathbb{R}} \|(i\beta - \mathcal{L})^{-1}\| < \infty, \tag{2.50}$$

where $\rho(\mathcal{L})$ denotes the resolvent set of the operator $\mathcal{L}$.

Our goal is to check that (2.50) does not hold. For that purpose, we introduce the positive self-adjoint operator $-\Delta_R$ defined by

$$D(-\Delta_R) = \{y \in H^2(\Omega) \cap V : \frac{\partial y}{\partial \nu} + \kappa y = 0 \text{ on } \Gamma_0\},$$

where $\kappa = (C, C) = C^* MC > 0$, and

$$-\Delta_R y = -\Delta y, \forall y \in D(-\Delta_R).$$

Denote by $\{\lambda_{R,k}^2\}_{k\in\mathbb{N}^*}$ the (discrete) spectrum of $-\Delta_R$ (repeated according to their multiplicity) and let y_k be the corresponding orthonormalized eigenvectors. Without loss of generality we can assume that the $\lambda_{R,k}$'s are positive. As $-\Delta_R$ has a compact resolvent, $\lambda_{R,k}$ goes to $+\infty$ as k goes to $+\infty$.

The index R was chosen because the boundary

$$\frac{\partial y}{\partial \nu} + \kappa y = 0 \text{ on } \Gamma_0$$

is of Robin type.

Recall also the following trace inequality from [20]

$$\int_{\Gamma_0} |u|^2 \, d\sigma \lesssim \|u\|_{L^2(\Omega)} \|u\|_V, \forall u \in V. \tag{2.51}$$

Indeed, it suffices to apply the standard trace theorem

$$\|v\|_{L^1(\Gamma_0)} \lesssim \|v\|_{W^{1,1}(\Omega)}, \forall v \in W^{1,1}(\Omega),$$

with $v = u^2$ to find

$$\int_{\Gamma_0} |u|^2 \, d\sigma \lesssim \|u^2\|_{L^1(\Omega)} + \|\nabla u^2\|_{L^1(\Omega)},$$

and by Leibniz's rule, Cauchy-Schwarz's inequality and Poincaré's inequality we obtain (2.51).

Proposition 2.5.2 *For all $k \in \mathbb{N}^*$, take $\mu_k = \lambda_{R,k}$. Then there exists a sequence of elements $U_k \in D(\mathcal{A})$ such that for all $k \in \mathbb{N}^*$:*

$$\|U_k\|_{\mathcal{H}} \geq 1, \tag{2.52}$$

$$\|(i\mu_k - \mathcal{A})U_k\|_{\mathcal{H}} \lesssim \mu_k^{-\frac{1}{2}}. \tag{2.53}$$

Proof. Fix an arbitrary $k \in \mathbb{N}^*$, then we define U_k as follows:

$$U_k = \mu_k^{-1}(y_k, i\mu_k y_k, \delta_k)^\top,$$

where

$$\delta_k = -Cy_k \text{ on } \Gamma_0$$

By this choice we check that U_k belongs to $D(\mathcal{A})$ because

$$(\delta_k, C) = -(Cy_k, C) = -\kappa y_k = \frac{\partial y_k}{\partial \nu} \text{ on } \Gamma_0.$$

First, by definition, we clearly have

$$\|U_k\|_{\mathcal{H}} \geq \|y_k\|_{L^2(\Omega)} = 1,$$

which proves (2.52). Now by construction, we see that

$$(i\mu_k - \mathcal{A})U_k = \mu_k^{-1}(0, 0, (i\mu_k - B)\delta_k + i\mu_k C y_k). \tag{2.54}$$

Moreover, we have

$$(i\mu_k - B)\delta_k + i\mu_k C y_k = BCy_k.$$

This identity in (2.54) yields

$$\|(i\mu_k - \mathcal{A})U_k\|_{\mathcal{H}}^2 = \mu_k^{-2} \int_{\Gamma_0} \|BCy_k\|_M^2 \, d\sigma \lesssim \mu_k^{-2} \int_{\Gamma_0} |y_k|^2 \, d\sigma.$$

Applying the trace estimate (2.51) we obtain

$$\|(i\mu_k - \mathcal{A})U_k\|_{\mathcal{H}}^2 \lesssim \mu_k^{-2}\|y_k\|_V \sim \mu_k^{-1}.$$

This proves (2.53). ■

Theorem 2.5.3 *The C_0-semigroup associated with $\mathcal{A}$ in $\mathcal{H}$ is not exponentially stable.*

Proof. The only non trivial case is the case when (2.49) holds for $\mathcal{A}$. In that case we need to show that (2.50) does not hold. Indeed, setting $\psi_k = (i\mu_k - \mathcal{A})U_k$ (that cannot be zero because $(i\mu_k - \mathcal{A})$ is invertible) we then have

$$\begin{aligned}
\|(i\mu_k - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} &= \sup_{\Psi \in \mathcal{H}, \Psi \neq 0} \frac{\|(i\mu_k - \mathcal{A})^{-1}\Psi\|_{\mathcal{H}}}{\|\Psi\|_{\mathcal{H}}} \\
&\geq \frac{\|(i\mu_k - \mathcal{A})^{-1}\Psi_k\|_{\mathcal{H}}}{\|\Psi_k\|_{\mathcal{H}}} \\
&\geq \frac{\|U_k\|_{\mathcal{H}}}{\|(i\mu_k - \mathcal{A})U_k\|_{\mathcal{H}}}.
\end{aligned}$$

Hence, by (2.52) and (2.53), we deduce that

$$\|(i\mu_k - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \gtrsim \sqrt{\mu_k},$$

which implies that (2.50) does not hold and the proof is thus complete. ■

2.6 Polynomial stability: a frequency domain approach

In this section we prove under some conditions the polynomial stability of the energy of the semigroup generated by $\mathcal{A}$.We use the following result from [19] (see also [14, 15]).

Theorem 2.6.1 *Let $(T(t))_{t\geq 0}$ be a bounded C_0-semigroup on a Hilbert space H with generator A such that $i\mathbb{R} \subset \rho(A)$. Then for a fixed $\alpha > 0$ the following conditions are equivalent:*
(i)

$$\|R(is, A)\| = O(|s|^{\alpha}), \; s \to \infty.$$

(ii)

$$\|T(t)A^{-1}\| = O(t^{-1/\alpha}), \; t \to \infty.$$

Consider the system

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) = 0 & , x \in \Omega, t > 0, \\ u(x,t) = 0 & , x \in \Gamma_1, t > 0, \\ \frac{\partial u}{\partial \nu}(x,t) = -u_t & , x \in \Gamma_0, t > 0, \end{cases} \tag{2.55}$$

Define the operator A corresponding to the system (2.55) by

$$AU = (v, \Delta u)^\top, U = (u,v)^\top \in D(A), \tag{2.56}$$

with

$$D(A) = \{U \in H : \Delta u \in L^2(\Omega), v \in V, \frac{\partial u}{\partial \nu} = -v \text{ on } \Gamma_0\},\ \ H = V \times L^2(\Omega).$$

In the next two propositions we use the exponential or polynomial stability of system (2.55) to prove a polynomial stability of system (2.2) with a certain decay rate depending on the type of stability of (2.55).

Proposition 2.6.2 *Assume that the energy of system (2.55) is exponentially stable and $i\mathbb{R} \subset \rho(\mathcal{A})$. Suppose moreover that there exist $p > 0$ and $\alpha > 0$ such that for $s \in \mathbb{R}$ with $|s|$ large enough we have*

$$\Re((isI - B)^{-1}C, C) \geq \frac{\alpha}{|s|^{2p}}, \tag{2.57}$$

then the energy of the solution of (2.2) satisfies the polynomial decay

$$E(t) \lesssim \frac{1}{t^{1/(p+1/2)}} \|U_0\|^2_{D(\mathcal{A})},\ \forall t > 0. \tag{2.58}$$

Proof. For $s \in \mathbb{R}$ and $F = (y_1, z_1, \delta_1)^\top \in \mathcal{H}$, let $U_F = (y,z,\delta)^\top = (isI - \mathcal{A})^{-1}F$. Then proceeding as in Proposition 2.2.1 but replacing λ by is in the equation (2.13), we obtain

$$a_{is}(y,\varphi) = L_F(\varphi), \forall \varphi \in V = \{y \in H^1(\Omega) : y = 0 \text{ on } \Gamma_1\}, \tag{2.59}$$

where the expressions of a_λ and L_F are respectively given by the identities (2.14) and (2.15). For $\varphi = y$, we find

$$\begin{aligned} &\int_\Omega -s^2 y\bar{y}dx + \int_\Omega \nabla y \nabla \bar{y} dx + \int_{\Gamma_0} is((isI - B)^{-1}C, C)\gamma_0 y \gamma_0 \bar{y} ds \\ &= \int_\Omega (z_1 + isy_1)\bar{y}dx + \int_{\Gamma_0} \left((isI - B)^{-1}(C\gamma_0 y_1 + \delta_1), C\right) \gamma_0 \bar{y} ds. \end{aligned} \tag{2.60}$$

Taking the imaginary part of (2.60), we get

$$\int_{\Gamma_0} s\Re((isI - B)^{-1}C, C)|\gamma_0 y|^2 ds = \Im\left(\int_\Omega (z_1 + isy_1)\bar{y}dx + \int_{\Gamma_0} \left((isI - B)^{-1}(C\gamma_0 y_1 + \delta_1), C\right)\gamma_0 \bar{y} ds\right). \tag{2.61}$$

Taking the modulus of (2.61) and using Cauchy-Schwarz's inequality, we obtain

$$\begin{aligned} |s| \int_{\Gamma_0} \Re((isI - B)^{-1}C, C)|\gamma_0 y|^2 ds &\leq \int_\Omega |(z_1 + isy_1)\bar{y}| dx + \int_{\Gamma_0} \left|\left((isI - B)^{-1}(C\gamma_0 y_1 + \delta_1), C\right)\gamma_0 \bar{y}\right| ds \\ &\lesssim \|F\|_{\mathcal{H}} \|y\|_{L^2(\Omega)} + |s| \|F\|_{\mathcal{H}} \|y\|_{L^2(\Omega)} + \frac{1}{|s|} \|F\|_{\mathcal{H}} \|y\|_{L^2(\Gamma_0)}. \end{aligned}$$

By (2.57) we deduce that for $|s|$ large enough,

$$|s|\int_{\Gamma_0}\frac{1}{|s|^{2p}}|\gamma_0 y|^2 ds \lesssim |s|\|F\|_{\mathcal{H}}\|y\|_{L^2(\Omega)}+\frac{1}{|s|}\|F\|_{\mathcal{H}}\|y\|_{L^2(\Gamma_0)},$$

thus

$$\|y\|^2_{L^2(\Gamma_0)} \lesssim |s|^{2p}\|F\|_{\mathcal{H}}\|y\|_{L^2(\Omega)}+|s|^{2p-2}\|F\|_{\mathcal{H}}\|y\|_{L^2(\Gamma_0)}, \tag{2.62}$$

hence by Young's inequality

$$\|y\|^2_{L^2(\Gamma_0)} \lesssim |s|^{2p}\|F\|_{\mathcal{H}}\|y\|_{L^2(\Omega)}+|s|^{2(2p-2)}\|F\|^2_{\mathcal{H}}. \tag{2.63}$$

For every $s \in \mathbb{R}$ and for every $f \in L^2(\Omega)$ let us show that there exists a solution $\varphi_f \in H^1(\Omega)$ of the problem

$$\begin{cases} -(s^2+\Delta)\varphi_f = f & , x\in\Omega, t>0,\\ \varphi_f(x,t)=0 & , x\in\Gamma_1, t>0,\\ \frac{\partial\varphi_f}{\partial\nu}(x,t) = -is\varphi_f & , x\in\Gamma_0, t>0, \end{cases} \tag{2.64}$$

and satisfying

$$\begin{cases} |s|\|\varphi_f\|_{L^2(\Omega)}+\|\varphi_f\|_{H^1(\Omega)} & \lesssim & \|f\|_{L^2(\Omega)},\\ |s|\|\varphi_f\|_{L^2(\Gamma_0)} & \lesssim & \|f\|_{L^2(\Omega)}. \end{cases} \tag{2.65}$$

Indeed, by Huang-Prüss Theorem (see [23, 26, 43]) the exponential stability of system (2.55) implies that there exists $M>0$ such that

$$\|(isI-A)^{-1}\|_{\mathcal{L}(\mathcal{H})} \le M < +\infty, \tag{2.66}$$

for all $s\in\mathbb{R}$. Due to (2.66) we have

$$\forall f\in L^2(\Omega),\ \forall s\in\mathbb{R},\ \exists! u_f = \begin{pmatrix}\varphi_f\\ \psi_f\end{pmatrix}\in D(A) \text{ s.t. } (isI-A)u_f = \begin{pmatrix}0\\ f\end{pmatrix}$$

and such that

$$\|u_f\|_H \le M\|f\|_{L^2(\Omega)}.$$

We deduce that

$$\begin{cases} is\varphi_f-\psi_f &=& 0\\ is\psi_f-\Delta\varphi_f &=& f\end{cases}$$

which gives $\psi_f = is\varphi_f$ and $(s^2+\Delta)\varphi_f = -f$. Moreover,

$$\|\varphi_f\|_{H^1(\Omega)}+|s|\|\varphi_f\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)},$$

and the first estimate of (2.65) hold. To obtain the third inequality we write

$$\int_\Omega(-s^2\varphi_f-f)\bar\varphi_f dx+\int_\Omega|\nabla\varphi_f|^2dx = \int_\Omega\Delta\varphi_f\bar\varphi_f dx+\int_\Omega|\nabla\varphi_f|^2dx = \int_{\Gamma_0}\partial_n\varphi_f\bar\varphi_f ds,$$

hence

$$\int_{\Gamma_0} \partial_n \varphi_f \bar{\varphi}_f ds = \int_\Omega (|\nabla \varphi_f|^2 - s^2|\varphi_f|^2 - f\bar{\varphi}_f)dx. \tag{2.67}$$

As $\psi_f = is\varphi_f$ and $\frac{\partial \varphi_f}{\partial \nu} = -\psi_f$, then taking the imaginary part of (2.67) we get

$$|s| \int_{\Gamma_0} |\varphi_f|^2 ds = |\Im \int_\Omega f\bar{\varphi}_f dx| \leq \|f\|_{L^2(\Omega)} \|\varphi_f\|_{L^2(\Omega)} \lesssim \frac{\|f\|^2_{L^2(\Omega)}}{|s|},$$

which proves the second estimate of (2.65).

We first show that

$$\|y\|_{L^2(\Omega)} + \|y\|_{L^2(\Gamma_0)} \lesssim |s|^{2p} \|F\|_{\mathcal{H}}. \tag{2.68}$$

By replacing φ by φ_f in the identity (2.13) and integrating by parts we get

$$\begin{aligned}\int_\Omega -s^2 y\bar{\varphi}_f dx - \int_\Omega y\Delta\bar{\varphi}_f dx + \int_{\Gamma_0} y\frac{\partial \bar{\varphi}_f}{\partial \nu} ds + \int_{\Gamma_0} is \left((isI - B)^{-1}C, C\right) \gamma_0 y \gamma_0 \bar{\varphi}_f ds. \\ = \int_\Omega (z_1 + isy_1)\bar{\varphi}_f dx + \int_{\Gamma_0} \left((isI - B)^{-1}(C\gamma_0 y_1 + \delta_1), C\right) \gamma_0 \bar{\varphi}_f ds.\end{aligned}$$

We deduce that

$$\begin{aligned}\int_\Omega y\bar{f}dx = \int_{\Gamma_0} isy\bar{\varphi}_f dx + \int_\Omega (z_1 + isy_1)\bar{\varphi}_f dx + \int_{\Gamma_0} \left((isI - B)^{-1}(C\gamma_0 y_1 + \delta_1), C\right) \gamma_0 \bar{\varphi}_f ds \\ - \int_{\Gamma_0} is \left((isI - B)^{-1}C, C\right) \gamma_0 y \gamma_0 \bar{\varphi}_f ds\end{aligned} \tag{2.69}$$

Take $f = y$ in (2.69) to obtain

$$\begin{aligned}\int_\Omega |y|^2 dx = \int_{\Gamma_0} isy\bar{\varphi}_y dx + \int_\Omega (z_1 + isy_1)\bar{\varphi}_y dx + \int_{\Gamma_0} \left((isI - B)^{-1}(C\gamma_0 y_1 + \delta_1), C\right) \gamma_0 \bar{\varphi}_y ds \\ - \int_{\Gamma_0} is \left((isI - B)^{-1}C, C\right) \gamma_0 y \gamma_0 \bar{\varphi}_y ds.\end{aligned}$$

Using Cauchy-Schwarz inequality together with (2.65), we obtain for $|s|$ large enough

$$\int_\Omega |y|^2 dx \lesssim \|y\|_{L^2(\Gamma_0)} \|y\|_{L^2(\Omega)} + \|F\|_{\mathcal{H}} \|y\|_{L^2(\Omega)}.$$

By Young's inequality, we deduce that

$$\|y\|^2_{L^2(\Omega)} \lesssim \|y\|^2_{L^2(\Gamma_0)} + \|F\|^2_{\mathcal{H}}. \tag{2.70}$$

Using this estimate in (2.63), we get

$$\|y\|^2_{L^2(\Gamma_0))} \lesssim |s|^{2p} \|F\|_{\mathcal{H}} (\|y\|_{L^2(\Gamma_0)} + \|F\|_{\mathcal{H}}) + |s|^{2(2p-2)} \|F\|^2_{\mathcal{H}},$$

and again by Young's inequality, we obtain

$$\|y\|^2_{L^2(\Gamma_0))} \lesssim |s|^{4p} \|F\|^2_{\mathcal{H}}.$$

By (2.70) we deduce that

$$\|y\|^2_{L^2(\Omega)} \lesssim |s|^{4p}\|F\|^2_{\mathcal{H}},$$

which proves (2.68). It follows that

$$\|z\|^2_{L^2(\Omega)} = \|isy - y_1\|^2_{L^2(\Omega)} \leq |s|^{4p+2}\|F\|^2_{\mathcal{H}}.$$

Moreover, by the expression (2.11) and (2.68), we get

$$\|\delta\|_{(L^2(\Gamma_0))^n} \lesssim \frac{1}{|s|}(\|F\|_{\mathcal{H}} + |s|\|y\|_{L^2(\Gamma_0)}) \lesssim |s|^{2p}\|F\|_{\mathcal{H}}.$$

We further have

$$\int_\Omega |\nabla y|^2 dx \lesssim |s|^{4p+2}\|F\|^2_{\mathcal{H}}.$$

Indeed, by (2.60), we have

$$\begin{aligned}
\int_\Omega |\nabla y|^2 dx &= \int_\Omega s^2|y|^2 dx - \int_{\Gamma_0} is((isI - B)^{-1}C, C)|\gamma_0 y|^2 ds + \int_\Omega (z_1 + isy_1)\bar{y} dx \\
&\quad + \int_{\Gamma_0} \big((isI - B)^{-1}(C\gamma_0 y_1 + \delta_1), C\big)\, \gamma_0 \bar{y} ds \\
&\lesssim |s|^2\|y\|^2_{L^2(\Omega)} + \|y\|^2_{L^2(\Gamma_0)} + |s|\|F\|_{\mathcal{H}}\|y\|_{L^2(\Omega)} + \frac{1}{|s|}\|F\|_{\mathcal{H}}\|y\|_{L^2(\Gamma_0)} \\
&\lesssim |s|^{4p+2}\|F\|^2_{\mathcal{H}} + |s|^{4p}\|F\|^2_{\mathcal{H}} + |s|^{2p+1}\|F\|^2_{\mathcal{H}} + |s|^{2p-1}\|F\|^2_{\mathcal{H}}.
\end{aligned}$$

Hence we have shown that

$$\|U_F\|_{\mathcal{H}} \lesssim |s|^{2p+1}\|F\|_{\mathcal{H}},$$

for $|s|$ large enough. This means that

$$\|(is - \mathcal{A})^{-1}\| = O(|s|^{2p+1}),\ s \to \infty,$$

and it follows by Theorem 2.6.1 that

$$\|U(t)\|_{\mathcal{H}} \lesssim \frac{1}{t^{1/(2p+1)}}\|U_0\|_{D(\mathcal{A})},$$

which proves (2.58). ∎

Proposition 2.6.3 *Suppose as in Proposition 2.6.2 that there exist $p > 0$ and $\alpha > 0$ such that for $s \in \mathbb{R}$ with $|s|$ large enough, (2.57) holds and that $i\mathbb{R} \subset \rho(\mathcal{A})$. Assume moreover that the energy of system (2.55) is polynomially stable with*

$$\|e^{At}A^{-1}\| = O(t^{-1/\alpha}),\ t \to \infty,$$

for some $\alpha > 0$. Then the energy of the solution of (2.2) satisfies a polynomial decay

$$E(t) \lesssim \frac{1}{t^{1/(p+\frac{\alpha+1}{2})}}\|U_0\|^2_{D(\mathcal{A})},\ \forall t > 0. \tag{2.71}$$

Proof. Proceeding as in the proof of Proposition 2.6.2, for $F \in \mathcal{H}$ and $U_F \in D(\mathcal{A})$ defined therein we obtain (2.60), (2.61), (2.62), and (2.63). In addition, there exists $\varphi_f \in V$ satisfying (2.64) and

$$\begin{cases} |s|\|\varphi_f\|_{L^2(\Omega)} + \|\varphi_f\|_{H^1(\Omega)} & \lesssim & |s|^\alpha \|f\|_{L^2(\Omega)}, \\ |s|\|\varphi_f\|_{L^2(\Gamma_0)} & \lesssim & |s|^{\frac{\alpha}{2}} \|f\|_{L^2(\Omega)}. \end{cases} \tag{2.72}$$

Indeed, since the energy of system (2.55) is polynomially stable then using Theorem 2.6.1,

$$\forall f \in L^2(\Omega),\ \forall s \in \mathbb{R},\ \exists! u_f = (\varphi_f, \psi_f)^\top \in D(A) \text{ s.t. } (isI - A)u_f = (0, f)^\top$$

with

$$\|u_f\|_H \leq M|s|^\alpha \|f\|_{L^2(\Omega)},$$

for some $M > 0$. As before, we deduce that

$$\begin{cases} \|y\|_{L^2(\Omega)} & \lesssim & |s|^{2p}|s|^\alpha \|F\|_{\mathcal{H}} \\ \|y\|_{L^2(\Gamma_0)} & \lesssim & |s|^{2p}|s|^{\frac{\alpha}{2}} \|F\|_{\mathcal{H}} \end{cases} \tag{2.73}$$

Replacing $f = y$ in (2.69) and using the estimates (2.72) together with Young's inequality, we obtain

$$\|y\|^2_{L^2(\Omega)} \lesssim |s|^\alpha \left(\|y\|^2_{L^2(\Gamma_0)} + |s|^\alpha \|F\|^2_{\mathcal{H}} \right). \tag{2.74}$$

Then by (2.63), we get

$$\|y\|^2_{L^2(\Gamma_0)} \lesssim |s|^{4p+\alpha} \|F\|^2_{\mathcal{H}}.$$

Due to (2.74),

$$\|y\|^2_{L^2(\Omega)} \lesssim |s|^{4p+2\alpha} \|F\|^2_{\mathcal{H}},$$

and thus

$$\|z\|^2_{L^2(\Omega)} = \|isy - y_1\|^2_{L^2(\Omega)} \leq |s|^{4p+2\alpha+2} \|F\|^2_{\mathcal{H}}.$$

By the expression (2.11) and (2.73), we get

$$\|\delta\|_{(L^2(\Gamma_0))^n} \lesssim \frac{1}{|s|} (\|F\|_{\mathcal{H}} + |s|\|y\|_{L^2(\Gamma_0)}) \lesssim |s|^{2p+\frac{\alpha}{2}} \|F\|_{\mathcal{H}}.$$

Using (2.60) and (2.73), we have

$$\int_\Omega |\nabla y|^2 dx \lesssim |s|^{4p+2\alpha+2} \|F\|^2_{\mathcal{H}}.$$

Henceforth,

$$\|(is - \mathcal{A})^{-1}\| = O(|s|^{2p+\alpha+1}),\ s \to \infty,$$

and the estimate (2.71) follows from Theorem 2.6.1. ■

Before going on, let us give sufficient conditions on B and C that guarantee that (2.57) holds.

Proposition 2.6.4 *Suppose that B, C and M have constant scalar entries. Let B^* be the adjoint of B with respect to $(\cdot,\cdot)_{\mathbb{C}^n}$, $B_0 = B - R$ and $R = \frac{B+B^*}{2}$. Moreover, assume that*

$$m = \min\{p \in \mathbb{N} : P(B_0^p C) \neq 0\} \tag{2.75}$$

exists, with P the projection from $\mathbb{C}^n$ into $(\ker R)^\perp$. Then there exists $\alpha > 0$ such that for $|s|$ large enough, we have

$$\Re((isI - B)^{-1}C, C) \geq \frac{\alpha}{|s|^{2(m+1)}}.$$

Proof. Indeed, we have $RB^lC = 0$ for all $l < m$ and $RB^mC = RB_0^mC$. Then, since R is self-adjoint and $RC = RPC$, we may write

$$-(R\sum_{j=0}^{\infty}(-i)^{j+1}\frac{B^j}{s^{j+1}}C,\sum_{l=0}^{\infty}(-i)^{l+1}\frac{B^l}{s^{l+1}}C) = -((-i)^{m+1}\frac{B^m}{s^{m+1}}C, R(-i)^{m+1}\frac{B^m}{s^{m+1}}C) + O(|s|^{-(2m+3)})$$
$$= -((-i)^{m+1}\frac{PB_0^m}{s^{m+1}}C, (-i)^{m+1}R\frac{PB_0^m}{s^{m+1}}C) + O(|s|^{-(2m+3)}).$$

As

$$\Re((isI-B)^{-1}C,C) = \Re((isI-B)^{-1}C,(isI-B)(isI-B)^{-1}C)$$
$$= -(R(isI-B)^{-1}C,(isI-B)^{-1}C) = -(R\sum_{j=0}^{\infty}(-i)^{j+1}\frac{B^j}{s^{j+1}}C,\sum_{l=0}^{\infty}(-i)^{l+1}\frac{B^l}{s^{l+1}}C),$$

we get

$$\Re((isI-B)^{-1}C,C) = -\frac{(RP(B_0^mC),P(B_0^mC))}{s^{2(m+1)}} + O(|s|^{-(2m+3)}).$$

But $-R$ defines a norm on $(\ker R)^{\perp}$, thus $-(RP(B_0^mC),P(B_0^mC)) \gtrsim \|P(B_0^mC)\|^2 > 0$. ∎

Remark 2.6.5 *The semigroup $e^{\mathcal{A}t}$ is exponentially stable for domains with smooth boundary (of class C^{∞}) satisfying the geometric control condition (G.C.C)(see [12]), as well as for domains of class C^2 satisfying the vector field assumptions described in [27] (see (i),(ii),(iii) of Theorem 1 in [27]). Moreover, in Theorem 1.2 of [28] the authors prove the exponential stability of $e^{\mathcal{A}t}$ for smooth domains under weaker geometric conditions than in [27] (without (ii) of Theorem 1).*

Remark 2.6.6 *Consider the system (2.55) on the square $[0,1]\times[0,1]$ and suppose that $\Gamma_0 = \{1\}\times[0,1]$. Then similarly as in [40] we deduce that e^{At} is polynomially stable with $\|e^{At}U_0\|^2 \lesssim \frac{1}{t}\|U_0\|^2_{D(A)}$, for all $U_0 \in D(A)$. Then if the assumptions of Proposition 2.6.3 are satisfied, then $\|e^{\mathcal{A}t}U_0\|^2 \lesssim t^{-\frac{1}{p+\frac{3}{2}}}\|U_0\|^2_{D(\mathcal{A})}$, for all $U_0 \in D(\mathcal{A})$.*

2.7 Dissipation on the whole boundary

Here we assume that $\Gamma_1 = \emptyset$. Thus $\Gamma = \Gamma_0 = \partial\Omega$ and system (2.2) becomes

$$\begin{cases} y_{tt}(x,t) - \Delta y(x,t) = 0 & ,x\in\Omega, t>0,\\ \frac{\partial y}{\partial\nu}(x,t) = (\delta(x,t),C) & ,x\in\Gamma_0, t>0,\\ \delta_t(x,t) = B\delta(x,t) - Cy_t(x,t) & ,x\in\Gamma_0, t>0.\end{cases} \tag{2.76}$$

Let $V = H^1(\Omega)$, then $\mathcal{H}$ is given by

$$\mathcal{H} = V \times L^2(\Omega)\times (L^2(\Gamma_0))^n,$$

and is now endowed with the following inner product

$$((y,z,\delta),(y_1,z_1,\delta_1))_{\mathcal{H}} = \int_{\Omega} y\bar{y}_1dx + \int_{\Omega}\nabla y\nabla\bar{y}_1dx + \int_{\Omega} z\bar{z}_1dx + \int_{\Gamma_0}(\delta,\delta_1)dx.$$

We define the operator $(\mathcal{A}, D(\mathcal{A}))$ on $\mathcal{H}$ as in chapter 1,

$$\mathcal{A}U = (z, \Delta y, B\delta - C\gamma_0 z)^\top, \text{ for } U = (y, z, \delta)^\top \in D(\mathcal{A}),$$

where $D(\mathcal{A}) = \{U \in \mathcal{H} : \Delta y \in L^2(\Omega), z \in V, \frac{\partial y}{\partial \nu} = C^\intercal M\delta \text{ on } \Gamma_0\}$. Remark that $\mathcal{A}$ is not dissipative with respect to the norm defined on $\mathcal{H}$. Indeed,

$$\Re(\mathcal{A}U, U)_{\mathcal{H}} = \int_{\Gamma_0} \Re(B\delta, \delta)_{\mathbb{R}^n} ds + \Re \int_\Omega z\bar{y} dx.$$

To overcome this difficulty we need further assumptions on B and C.
We actually suppose that B is invertible on the whole boundary and consider the following cases: either $\int_{\Gamma_0}((-B)^{-1}C, C)ds \neq 0$ or $\int_{\Gamma_0}((-B)^{-1}C, C)ds = 0$.

Before going on, recall that L_F defined by (2.15) with $\lambda = 0$ is given by

$$L_F(\varphi) = \int_\Omega z\bar{\varphi} dx + \int_\Gamma \left((-B)^{-1}(Cy + \delta), C\right) \gamma_0 \bar{\varphi} ds, \forall \varphi \in H^1(\Omega),$$

when $F = (y, z, \delta)^\top \in \mathcal{H}$. For shortness, we further set

$$L_1(F) = L_F(1).$$

Note that

$$L_1(\mathcal{A}U) = 0, \forall U \in D(\mathcal{A}).$$

2.7.1 The first case

Throughout the remainder of this subsection, B is assumed to be invertible on the whole boundary and $\int_\Gamma (B^{-1}C, C)ds$ is nonzero. Then we introduce the following subspace $\widetilde{\mathcal{H}}$ of $\mathcal{H}$:

$$\widetilde{\mathcal{H}} = \{F \in \mathcal{H} : L_1(F) = 0\},$$

endowed with the inner product (2.4), and the operator $(\widetilde{\mathcal{A}}, D(\widetilde{\mathcal{A}}))$ defined by

$$D(\widetilde{\mathcal{A}}) = \widetilde{\mathcal{H}} \cap D(\mathcal{A}), \ \widetilde{\mathcal{A}}U = \mathcal{A}U, \ \forall U \in D(\widetilde{\mathcal{A}}). \tag{2.77}$$

Proposition 2.7.1 *The operator $\widetilde{\mathcal{A}}$ is m-dissipative.*

Proof. The dissipativity of $\widetilde{\mathcal{A}}$ directly follows from the property (2.9). Proceeding as in the proof of Proposition 2.2.1, we apply Lax-Milgram lemma to get the existence of a unique solution $y \in H^1(\Omega)$ of (2.13), and deduce the surjectivity of $\lambda I - \mathcal{A}$ from $D(\mathcal{A})$ onto $\mathcal{H}$ for all $\lambda > 0$. Moreover, we can easily check that the preimage $U = (y, z, \delta)^\top$ of $F = (y_1, z_1, \delta_1)^\top \in \widetilde{\mathcal{H}}$ by $\lambda I - \mathcal{A}$ is also in $\widetilde{\mathcal{H}}$, thus proving that $\lambda I - \widetilde{\mathcal{A}}$ is surjective from $D(\widetilde{\mathcal{A}})$ onto $\widetilde{\mathcal{H}}$. ■

Proposition 2.7.2 *The operator $\widetilde{\mathcal{A}}$ is one-to-one and onto.*

Proof. We first show that $0 \notin \sigma_d(\widetilde{\mathcal{A}})$. Suppose that $\widetilde{\mathcal{A}}(y, z, \delta)^\top = (z, \Delta y, B\delta - Cz)^\top = 0$ for some $(y, z, \delta)^\top \in D(\widetilde{\mathcal{A}})$, then $z = 0$ and as B is invertible $\delta = 0$ thus $\partial_n y = 0$. But $\Delta y = 0$ in Ω, then multiplying by y and integrating by parts we deduce that y is constant in Ω. Since

$(y, z, \delta)^\top \in \widetilde{\mathcal{H}}$, we get $\int_{\Gamma_0}(B^{-1}C, C)yds = 0$ and we conclude that $y = 0$.

Let us now show that $\widetilde{\mathcal{A}}$ is surjective from $D(\widetilde{\mathcal{A}})$ onto $\widetilde{\mathcal{H}}$. Let $F = (y_1, z_1, \delta_1)^\top \in \widetilde{\mathcal{H}}$. Then F satisfies $L_F(1) = 0$ and it follows that for all $\dot{\varphi} \in H^1(\Omega)/\mathbb{C}$ the expression $L_F(\dot{\varphi}) = L_F(\varphi)$, with $\varphi \in \dot{\varphi}$ is a well defined linear bounded form on the quotient space $H^1(\Omega)/\mathbb{C}$ endowed with the norm $\|\dot{\varphi}\| = \|\nabla\varphi\|_{L^2(\Omega)}$. We also define the form a on $H^1(\Omega)/\mathbb{C}\times H^1(\Omega)/\mathbb{C}$ by $a(\dot{y}, \dot{\varphi}) = a_0(y, \varphi)$, $y \in \dot{y}, \varphi \in \dot{\varphi}$. Moreover, a being coercive on $H^1(\Omega)/\mathbb{C}$, Lax-Milgram lemma then implies the existence of a unique solution $\dot{y} \in H^1(\Omega)/\mathbb{C}$ of

$$a(\dot{y}, \dot{\varphi}) = L_F(\dot{\varphi}),\ \forall \dot{\varphi} \in H^1(\Omega)/\mathbb{C}.$$

Choose any $y \in \dot{y}$, then it satisfies

$$\int_\Omega \nabla y \nabla \bar{\varphi} dx = \int_\Omega z_1 \bar{\varphi} dx + \int_{\Gamma_0} \big((-B)^{-1}(Cy_1 + \delta_1), C\big)\, \bar{\varphi} ds, \forall \varphi \in H^1(\Omega). \tag{2.78}$$

In particular, for $\varphi \in D(\Omega)$ we get $-\Delta y = z_1 \in L^2(\Omega)$. Moreover, by replacing z_1 by $-\Delta y$ in (2.78), we deduce that $\partial_\nu y = ((-B)^{-1}(Cy_1+\delta_1), C)$. Set $\tilde{\delta} = B^{-1}(Cy_1+\delta_1)$, $\tilde{z} = y_1, \tilde{y} = -y+\beta$, with $\beta \in \mathbb{C}$ fixed below. Thus $B\tilde{\delta} - C\tilde{z} = \delta_1$ and

$$\Delta\tilde{y} = -\Delta y = z_1 \quad \text{in } \Omega,$$

and

$$\frac{\partial\tilde{y}}{\partial\nu} = -\frac{\partial y}{\partial\nu} = (\tilde{\delta}, C) \quad \text{on } \Gamma.$$

Since $\int_\Gamma (B^{-1}C, C)ds \neq 0$, we may choose

$$\beta = \frac{1}{\int_\Gamma (B^{-1}C, C)ds}\left(\int_\Omega y_1 dx + \int_\Gamma \left(B^{-1}\left(Cy - B^{-1}(Cy_1 + \delta_1)\right), C\right) ds\right)$$

to get

$$\int_\Omega \tilde{z} dx + \int_\Gamma ((-B)^{-1}(C\tilde{y} + \tilde{\delta}), C) ds = 0.$$

Henceforth, $\widetilde{U} = (\tilde{y}, \tilde{z}, \tilde{\delta})^\top \in D(\widetilde{\mathcal{A}})$ and $\widetilde{\mathcal{A}}\widetilde{U} = F$. ■

Corollary 2.7.3 $0 \in \sigma_d(\mathcal{A})$ *with multiplicity 1, its associated eigenvector is* $(1, 0, 0)^\top$.

Proof. Clearly $\mathcal{A}(y, z, \delta)^\top = (0, 0, 0)^\top$ if and only if y is constant in Ω, $z = 0$ and $\delta = 0$, hence $0 \in \sigma_d(\mathcal{A})$ with a geometric multiplicity equal to 1. As

$$L_1((1, 0, 0)^\top) = -\int_\Gamma (B^{-1}C, C)ds \neq 0,$$

$(1, 0, 0)^\top$ does not belong to the range of $\mathcal{A}$ and the algebraic multiplicity of 0 is also equal to 1. ■

2.7.2 The second case

Throughout the remainder of this subsection, B is assumed to be invertible on the whole boundary and $\int_{\Gamma_0}(B^{-1}C, C)ds = 0$. For simplicity we further assume that B and C have real entries. In this setting as $(B^{-1}C, C) \leq 0$, we deduce that

$$(B^{-1}C, C) = 0 \text{ on } \Gamma. \tag{2.79}$$

Inspired by Example 1 below, we also suppose that

$$|\Omega| - \int_\Gamma (B^{-2}C, C)\, ds \neq 0. \tag{2.80}$$

Under these assumptions, we see that L_F and L_1 defined before are reduced to

$$L_F(\varphi) = \int_\Omega z\bar{\varphi}dx + \int_\Gamma \big((-B)^{-1}\delta, C\big)\, \gamma_0\bar{\varphi}ds, \forall \varphi \in H^1(\Omega),$$

and

$$L_1(F) = \int_\Omega zdx - \int_\Gamma \big(B^{-1}\delta, C\big)\, ds,$$

for $F = (y, z, \delta) \in \mathcal{H}$. We also introduce the functional

$$L_2(F) = \int_\Omega ydx - \int_\Gamma \big(B^{-2}(C\gamma_0 y + \delta), C\big)\, ds,$$

when $F = (y, z, \delta) \in \mathcal{H}$. Note that

$$L_2(\mathcal{A}U) = L_1(U), \forall U \in D(\mathcal{A}),$$

and that our assumption (2.80) implies that $L_2((1,0,0)^\top) \neq 0$.

Now we introduce the following subspace $\widetilde{\mathcal{H}}$ of $\mathcal{H}$:

$$\widetilde{\mathcal{H}} = \{U \in \mathcal{H} : L_1(U) = L_2(U) = 0\},$$

still endowed with the inner product (2.4) and the operator $(\widetilde{\mathcal{A}}, D(\widetilde{\mathcal{A}}))$ defined by (2.77).

Proposition 2.7.4 *The operator $\widetilde{\mathcal{A}}$ is m-dissipative.*

Proof. We proceed using the same proof as that of Proposition 2.7.1. Noting in addition that $L_2(F) = 0$ and $L_1(U) = 0$ imply that $L_2(U) = 0$, the proof is thus complete. ■

Proposition 2.7.5 *The operator $\widetilde{\mathcal{A}}$ is one-to-one and onto.*

Proof. Suppose that $\widetilde{\mathcal{A}}(y, z, \delta)^\top = (z, \Delta y, B\delta - Cz)^\top = 0$ for some $(y, z, \delta)^\top \in D(\widetilde{\mathcal{A}})$. Then, as before, we get $(y, z, \delta)^\top = c(1, 0, 0)^\top$ for some constant c. Since $(y, z, \delta)^\top \in \widetilde{\mathcal{H}}$, we conclude that $c = 0$. Therefore, $0 \notin \sigma_d(\widetilde{\mathcal{A}})$.

Let us now show that $\widetilde{\mathcal{A}}$ is surjective from $D(\widetilde{\mathcal{A}})$ onto $\widetilde{\mathcal{H}}$. Let $F = (y_1, z_1, \delta_1)^\top \in \widetilde{\mathcal{H}}$. Then F satisfies $L_F(1) = 0$ and as before there exists a solution $y \in H^1(\Omega)$ of (2.78). By setting $\tilde{\delta} = B^{-1}(Cy_1 + \delta_1)$, $\tilde{z} = y_1, \tilde{y} = -y + \beta$, with $\beta \in \mathbb{C}$ fixed below, we find that

$$\mathcal{A}(\tilde{y}, \tilde{z}, \tilde{\delta}) = F.$$

Furthermore, the assumption $L_2(F) = 0$ guarantees that $L_1(\tilde{y}, \tilde{z}, \tilde{\delta}) = 0$, while β is fixed in such a way that

$$L_2((\tilde{y}, \tilde{z}, \tilde{\delta})^\top) = L_2((-y, \tilde{z}, \tilde{\delta})^\top) + \beta L_2((1, 0, 0)^\top) = 0.$$

■

Corollary 2.7.6 *$0 \in \sigma_d(\mathcal{A})$ with geometric multiplicity 1 and algebraic multiplicity 2, its associated eigenvector is $(1, 0, 0)^\top$ and its generalized eigenvector is $(0, 1, B^{-1}C)^\top$.*

Proof. The conclusion follows from the fact that

$$\mathcal{A}(0, 1, B^{-1}C)^\top = (1, 0, 0)^\top,$$

and that

$$L_1((0, 1, B^{-1}C)^\top) = L_2((1, 0, 0)^\top) \neq 0.$$

■

Remark 2.7.7 *If (2.80) does not hold, then it equivalently means that $L_1((0, 1, B^{-1}C)^\top) = 0$, and in that case, we directly deduce that the algebraic multiplicity of 0 is greater than 3. For shortness, we let the remaining analysis to the reader.*

2.7.3 General considerations

From now on, we suppose that we are either in the first case or in the second one.

Proposition 2.7.8 *Suppose that the assumptions $(A_1), (A_2), (A_3)$ and (A_4) of Proposition 2.4.2 hold. Then $\sigma_d(\mathcal{A}) \cap i\mathbb{R}^* = \emptyset$. Consequently, $\sigma_d(\widetilde{\mathcal{A}}) \cap i\mathbb{R}^* = \emptyset$.*

Proof. Same proof as Proposition 2.4.2. ■

Proposition 2.7.9 *If $\lambda \notin \Sigma$, then $\lambda - \mathcal{A}$ is a Fredholm operator of index zero.*

Proof. Same proof as that of point (ii) of Theorem 2.3.1 but replacing V by $H^1(\Omega)$. ■

Remark 2.7.10 *If $\lambda \in \Sigma$, then $\lambda - \widetilde{\mathcal{A}}$ is singular. Indeed, since $\lambda - \mathcal{A}$ is singular then by Theorem 2.3.1, there exists a sequence $(U_n)_{n \in \mathbb{N}^*}$ in $D(\mathcal{A})$ satisfying $\|U_n\|_{\mathcal{H}} = 1$ and*

$$(\lambda I - \mathcal{A})U_n \to 0, \text{ as } n \to \infty.$$

In the first case, let $\alpha_n = \frac{L_1(U_n)}{L_1((1,0,0)^\top)}$, where $U_n = (y_n, z_n, \delta_n)^\top$. We now construct the sequence $(\widetilde{U}_n)_{n \in \mathbb{N}^*}$ by

$$\widetilde{U}_n = U_n - \alpha_n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

In the second case, let

$$\beta_n = \frac{L_1(U_n)}{L_1((0,1,B^{-1}C)^\top)}$$

and

$$\alpha_n = \frac{L_2(U_n) - \beta_n L_2((0,1,B^{-1}C)^\top)}{L_2((1,0,0)^\top)}.$$

In this case, we introduce the sequence $(\widetilde{U}_n)_{n\in\mathbb{N}^*}$ defined by

$$\widetilde{U}_n = U_n - \alpha_n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \beta_n \begin{pmatrix} 0 \\ 1 \\ B^{-1}C \end{pmatrix}.$$

Then $(\widetilde{U}_n)_n \subset D(\widetilde{A})$ and $\alpha_n \to 0, \beta_n \to 0$ as $n \to \infty$ due to Lemma 3.3 and the property (2.9). Consequently, $(\lambda I - \widetilde{\mathcal{A}})\widetilde{U}_n \to 0$ and $\|\widetilde{U}_n\|_{\widetilde{\mathcal{H}}} \to 1$. Therefore, taking $V_n = \frac{\widetilde{U}_n}{\|\widetilde{U}_n\|_{\widetilde{\mathcal{H}}}}$, we obtain $(V_n)_n \subset D(\widetilde{A})$ satisfying $\|V_n\|_{\widetilde{\mathcal{H}}} = 1$ and

$$(\lambda I - \mathcal{A})V_n \to 0, \text{ as } n \to \infty.$$

Proposition 2.7.11 *If (A_1) to (A_4) of Proposition 2.4.2 hold and if $\Sigma \cap i\mathbb{R}$ is countable, then the C_0-semigroup associated with $\widetilde{\mathcal{A}}$ is strongly stable.*

Proof. Due to Proposition 2.7.2 and Proposition 2.7.5, we have $0 \in \rho(\widetilde{\mathcal{A}})$. Now, let $i\lambda \in \mathbb{C} \setminus \Sigma$ with λ nonzero, then due to Proposition 2.7.8 we have $i\lambda \notin \sigma_d(\mathcal{A})$. We deduce by Proposition 2.7.9 that $i\lambda \in \rho(\mathcal{A})$. Using the same proof of the invertibility of $\lambda I - \widetilde{A}$ where $\lambda > 0$ of Proposition 2.7.1 we prove the invertibility of $i\lambda - \widetilde{A}$, and so $i\lambda \in \rho(\widetilde{A})$, hence

$$\mathbb{C} \setminus \Sigma \cap i\mathbb{R}^* \subset \rho(\widetilde{A}) \cap i\mathbb{R}^*$$

and thus

$$\sigma(\widetilde{A}) \cap i\mathbb{R}^* \subset \Sigma \cap i\mathbb{R}^*.$$

■

The next corollary follows directly from the previous Proposition.

Corollary 2.7.12 *Let the assumptions of Proposition 2.4.2 be satisfied. Suppose moreover that $\Sigma \cap i\mathbb{R}^* = \emptyset$. Then $i\mathbb{R}^* \subset \rho(\widetilde{A})$*

We define the operator $(A, D(A))$ on $H := H^1(\Omega) \times L^2(\Omega)$ as in the previous section. Moreover, we define the following subspace $\dot{H}$ of H

$$\dot{H} = \{U \in H : \int_\Omega z dx + \int_\Gamma y ds = 0\}$$

endowed with the inner product

$$((y, z), (y_1, z_1))_{\dot{H}} = \int_\Omega \nabla y \nabla \bar{y}_1 dx + \int_\Omega z \bar{z}_1 dx,$$

and the operator $(\dot{A}, D(\dot{A}))$ defined by

$$D(\dot{A}) = \dot{H} \cap D(A), \quad \dot{A}U = AU, \forall U \in D(\dot{A}).$$

We further define the following initial value problem on $\dot{H}$

$$U_t = \dot{A}U, \, U(0) = U_0. \tag{2.81}$$

Proposition 2.7.13 *Assume that $i\mathbb{R} \subset \rho(\widetilde{A})$ and that the energy of system (2.81) is exponentially stable. Suppose moreover that there exist $p > 0$ and $\alpha > 0$ such that for $s \in \mathbb{R}$ with $|s|$ large enough we have*

$$\Re((isI - B)^{-1}C, C) \geq \frac{\alpha}{|s|^{2p}},$$

then the energy of the solution of $U_t = \widetilde{\mathcal{A}}U, U(0) = U_0 \in D(\widetilde{\mathcal{A}})$, satisfies a polynomial decay

$$E(t) \lesssim \frac{1}{t^{1/(p+1/2)}} \|U_0\|^2_{D(\widetilde{\mathcal{A}})}, \, \forall t > 0. \tag{2.82}$$

Proof. For $f \in L^2(\Omega)$, we have $\begin{pmatrix} -\frac{1}{|\partial\Omega|}\int_\Omega f\,dx \\ f \end{pmatrix} \in \dot{H}$. Then there exists $\begin{pmatrix} \hat{u}_0 \\ \hat{u}_1 \end{pmatrix} \in D(\dot{A})$ satisfying

$$(\lambda I - \dot{A}) \begin{pmatrix} \hat{u}_0 \\ \hat{u}_1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{|\partial\Omega|}\int_\Omega f\,dx \\ f \end{pmatrix}$$

and

$$\|(\hat{u}_0, \hat{u}_1)^\top\|_{\dot{H}} \leq \|(-\frac{1}{|\partial\Omega|}\int_\Omega f\,dx, f)^\top\|_{\dot{H}}.$$

Take

$$u_0 = \hat{u}_0 + \frac{1}{is|\partial\Omega|}\int_\Omega f\,dx, u_1 = \hat{u}_1.$$

Then

$$(\lambda I - A) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

Setting $\varphi_f = u_0$, we get (2.65). Following the same proof of Proposition 2.6.2 we obtain the same estimates on the resolvent in the space $\widetilde{\mathcal{H}}$. We conclude by Theorem 2.6.1 as $i\mathbb{R} \subset \rho(\widetilde{\mathcal{A}})$. ■

Remark 2.7.14 *The semigroup $e^{\dot{A}t}$ is exponentially stable if the boundary Γ is smooth enough as it then satisfies the geometric control condition (G.C.C)(see [12]), or if Ω is of class C^2 satisfying the vector field assumptions described in [27] (more precisely (ii) and (iii) of Theorem 1 in [27]).*

Proposition 2.7.13 finally allows to obtain the asymptotic behavior of the solution of (2.76). To state properly the result, we again need to distinguish between the first and the second case.

Proposition 2.7.15 *Let the assumptions of Proposition 2.7.13 be satisfied, then the following statements hold:*
1. In the setting of case 1, the solution $U = (y, y_t, \delta)^\top$ of (2.76) with $U_0 \in D(\mathcal{A})$ satisfies

$$\|U(t) - \alpha(1, 0, 0)^\top\|^2_{\mathcal{H}} \lesssim \frac{1}{t^{1/(p+1/2)}} \|U_0\|^2_{D(\mathcal{A})},$$

where

$$\alpha = \frac{L_1(U_0)}{L_1((1,0,0)^\top)}.$$

2. In the setting of case 2, the solution $U = (y, y_t, \delta)^\top$ *of (2.76) with* $U_0 = (y_0, y_1, \delta_0)^\top \in D(\mathcal{A})$ *satisfies*

$$\|U(t) - \alpha(1,0,0)^\top - \beta(t, 1, B^{-1}C)^\top\|^2_{\mathcal{H}} \lesssim \frac{1}{t^{1/(p+1/2)}}\|U_0\|^2_{D(\mathcal{A})},$$

where

$$\beta = \frac{L_1(U_0)}{L_1((0,1,B^{-1}C)^\top)} \text{ and } \alpha = \frac{L_2(U_0) - \beta L_2((0,1,B^{-1}C)^\top)}{L_2((1,0,0)^\top)}.$$

Proof. In the first case, given $U_0 \in D(\mathcal{A})$ we set

$$\widetilde{U}_0 = U_0 - \alpha(1,0,0)^\top,$$

with α defined above. The choice of α guarantees that $\widetilde{U}_0$ belongs to $D(\widetilde{\mathcal{A}})$. Hence applying Proposition 2.7.13, we see that the solution $\widetilde{U}$ of

$$\widetilde{U}_t = \widetilde{\mathcal{A}}\widetilde{U}, \quad \widetilde{U}(0) = \widetilde{U}_0, \tag{2.83}$$

satisfies (2.82). The conclusion follows by noticing that

$$U(t) = \widetilde{U}(t) + \alpha(1,0,0)^\top,$$

is solution of (2.7),

$$U_t = \mathcal{A}U, \quad U(0) = U_0.$$

We proceed similarly in the second case, namely we set

$$\widetilde{U}_0 = U_0 - \alpha(1,0,0)^\top - \beta(0,1,B^{-1}C)^\top,$$

α and β being chosen such that $L_1(\widetilde{U}_0) = L_2(\widetilde{U}_0) = 0$. As before the solution $\widetilde{U}$ of (2.83) satisfies (2.82), and the conclusion follows by noticing that

$$U(t) = \widetilde{U}(t) + \alpha(1,0,0)^\top + \beta(t,1,B^{-1}C)^\top,$$

is solution of (2.7). ■

2.8 Polynomial stability: the multiplier method and the energy method

In this section we assume that

$$\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \phi, \text{ and } \text{meas } \Gamma_1 \neq 0,$$

and that there exists a point $x_0 \in \mathbb{R}^n$ such that

$$\Gamma_1 = \{x \in \Gamma | (x - x_0).\nu \leq 0\},$$

$$\Gamma_0 = \{x \in \Gamma | (x - x_0).\nu \geq a > 0\},$$

for some constant $a > 0$, where $\nu = \nu(x)$ denotes the unit outward normal vector at $x \in \Gamma$. We further suppose that B and C have constant and real entries (for variable entries, see Remark 2.8.10). Under these assumptions we are able to give a better energy decay rate than that given in the previous section once imposing certain algebraic conditions on B and C. Remark also that those geometrical conditions on the boundary exclude simply connected domains. Moreover, we set $q(x) = (x - x_0)$, for all $x \in \bar{\Omega}$ and denote by $R_0 = \|q\|_\infty = \sup\limits_{x\in\Omega}\|q(x)\|$.

We first prove a result from interpolation theory which later will allow us to deduce from the polynomial decay for certain regular initial data a polynomial decay of energy for less regular initial data (see also Proposition 3.1 in [14]).

Proposition 2.8.1 *Let $(A, D(A))$ be a m-dissipative generator of a C_0-semigroup of contractions on H, with A^{-1} bounded in a Hilbert space H with norm $\|\cdot\|$, let $U_0 \in D(A)$ and m be a positive integer. Suppose that $\|e^{At}V_0\|^2 \lesssim \frac{1}{t}\sum\limits_{j=0}^{m}\|A^jV_0\|^2$ for all $V_0 \in D(A^m)$, then*

$$\|e^{At}U_0\|^2 \lesssim \frac{1}{t^{\frac{1}{m}}}(\|U_0\|^2 + \|AU_0\|^2). \tag{2.84}$$

Proof. For $m = 1$, the result is clear.
For $m > 1$, applying Theorem 2.6 of [33] we get

$$\|e^{At}\|_{\mathcal{L}([H,D(A^m)]_\theta;H))} \leq \|e^{At}\|^{1-\theta}_{\mathcal{L}(H)}\|e^{At}\|^{\theta}_{\mathcal{L}(D(A^m);H))}, \ \forall\theta \in (0,1).$$

Due to Corollary 4.30 of [33], we have

$$[H, D(A^m)]_{\frac{1}{m}} = D(A).$$

By the assumptions we have $\|e^{At}\|_{\mathcal{L}(D(A^m);H))} \leq \frac{C}{t^{\frac{1}{2}}}$. Moreover, e^{At} is a family of contractions on H, thus $\|e^{At}\|_{\mathcal{L}(H)} \leq 1$ and hence

$$\|e^{At}\|_{\mathcal{L}(D(A);H)} \leq \|e^{At}\|^{\frac{1}{m}}_{\mathcal{L}(D(A^m);H))} \leq \frac{1}{t^{\frac{1}{2m}}},$$

which yields (2.84). ■

For $j \in \mathbb{N}$ and $U_0 \in D(\mathcal{A}^j)$, by Remark 2.2.4, for all $t \geq 0$, we can define $E_j(t) = \frac{1}{2}\|\partial_t^j U(t)\|^2_{\mathcal{H}}$. For $U_0 \in D(\mathcal{A}^{j+1})$, computing its derivative we obtain

$$\frac{d}{dt}E_j(t) = (\mathcal{A}\partial_t^j U, \partial_t^j U) = \int_{\Gamma_0}(B\partial_t^j\delta, \partial_t^j\delta)ds = \int_{\Gamma_0}\partial_t^j\delta^\intercal MB\partial_t^j\delta ds, \tag{2.85}$$

that is non positive. Hence this energy is also non increasing.

Proposition 2.8.2 *For $(y_0, z_0, \delta_0)^\intercal \in D(\mathcal{A})$, denote by (y, y_t, δ) the solution of system (2.2) and define $F_0(t)$ by*

$$F_0(t) = \int_\Omega\left(y_t q\cdot\nabla y + \frac{d-1}{2}yy_t\right)dx,$$

where we recall that $q(x) = x - x_0$. Then the derivative of F_0 is given by

$$\begin{aligned}\frac{d}{dt}F_0(t) &= -\frac{1}{2}\int_\Omega (|y_t|^2 + |\nabla y|^2)dx - \frac{1}{2}\int_{\Gamma_0} q\cdot\nu|\nabla y|^2 ds \\ &+\int_{\Gamma_0} C^\intercal M\delta q\cdot\nabla y ds + \frac{1}{2}\int_{\Gamma_1} q\cdot\nu|\nabla y|^2 ds \\ &+\frac{1}{2}\int_{\Gamma_0} q\cdot\nu|y_t|^2 ds + \frac{d-1}{2}\int_{\Gamma_0} C^\intercal M\delta y ds.\end{aligned}$$

Proof. We have

$$\begin{aligned}\frac{d}{dt}F_0(t) &= \frac{d}{dt}\int_\Omega y_t q\cdot\nabla y dx + \frac{d-1}{2}\frac{d}{dt}\int_\Omega y y_t dx \\ &= \int_\Omega y_{tt} q\cdot\nabla y dx + \int_\Omega y_t q\cdot\nabla y_t dx + \frac{d-1}{2}\left(\int_\Omega y y_{tt} dx + \int_\Omega |y_t|^2 dx\right).\end{aligned}$$

As $y_{tt} = \Delta y$, and using Green's formula we get

$$\begin{aligned}\int_\Omega y_{tt} q\cdot\nabla y dx &= \int_\Omega \Delta y q\cdot\nabla y dx = \int_\Gamma \frac{\partial y}{\partial\nu}\nabla y\cdot q ds - \int_\Omega \nabla y\cdot\nabla(q\cdot\nabla y)dx \\ &= \int_{\Gamma_0}\frac{\partial y}{\partial\nu}\nabla y\cdot q ds + \int_{\Gamma_1}(\nabla y\cdot\nu)(\nabla y\cdot q)ds - \int_\Omega \sum_{i=1}^d \frac{\partial y}{\partial x_i}\frac{\partial}{\partial x_i}\left(\sum_{j=1}^d (x_j - x_0^j)\frac{\partial y}{\partial x_j}\right)dx \\ &= \int_{\Gamma_0}\frac{\partial y}{\partial\nu}\nabla y\cdot q ds + \int_{\Gamma_1}|\nabla y|^2\nu\cdot q ds - \int_\Omega |\nabla y|^2 - \frac{1}{2}\int_\Omega q\cdot\nabla|\nabla y|^2 dx.\end{aligned}$$

Using Green's formula and as $y_t = 0$ on Γ_1, we get

$$\int_\Omega y_t q\cdot\nabla y_t dx = \frac{1}{2}\int_\Omega q\cdot\nabla|y_t|^2 dx = -\frac{1}{2}\int_\Omega \mathrm{div} q|y_t|^2 dx + \frac{1}{2}\int_{\Gamma_0} q\cdot\nu|y_t|^2 ds,$$

similarly substituting $\mathrm{div}q$ by d, we obtain

$$\int_\Omega \frac{1}{2}q\cdot\nabla|\nabla y|^2 dx = -\frac{d}{2}\int_\Omega |\nabla y|^2 dx + \frac{1}{2}\int_{\Gamma_0} q\cdot\nu|\nabla y|^2 ds + \frac{1}{2}\int_{\Gamma_1} q\cdot\nu|\nabla y|^2 ds. \tag{2.86}$$

Moreover, as $y = 0$ on Γ_1, we have

$$\int_\Omega y y_{tt} dx = \int_\Omega y\Delta y dx = -\int_\Omega |\nabla y|^2 dx + \int_{\Gamma_0}\frac{\partial y}{\partial\nu} y ds.$$

Finally, by replacing (2.86) in the expression of $\int_\Omega y_{tt} q\cdot\nabla y dx$ and summing all the above equalities, we obtain

$$\begin{aligned}\frac{d}{dt}F_0(t) &= \left(\frac{d-1}{2} - \frac{d}{2}\right)\int_\Omega |y_t|^2 dx + \left(\frac{d}{2} - 1 - \frac{d-1}{2}\right)\int_\Omega |\nabla y|^2 dx - \frac{1}{2}\int_{\Gamma_0} q.\nu|\nabla y|^2 ds \\ &+\int_{\Gamma_0}\frac{\partial y}{\partial\nu} q.\nabla y ds + \frac{1}{2}\int_{\Gamma_1} q.\nu|\nabla y|^2 ds \\ &+\frac{1}{2}\int_{\Gamma_0} q.\nu|y_t|^2 dx + \frac{d-1}{2}\int_{\Gamma_0}\frac{\partial y}{\partial\nu} y ds.\end{aligned}$$

Just substituting $\frac{\partial y}{\partial\nu} = C^\intercal M\delta$, we get the required result. ■

Proposition 2.8.3 *Suppose that there exist $k > 0, k_1 > 0$, a matrix $Q \in M_n(\mathbb{R})$, and a positive integer m such that $RB^qC = 0$ for all $0 \leq q \leq m-2$ and*

$$(C^\intercal(B\delta - Cv))^2 \leq k \left[\|\delta\|^2 - \sum_{j=1}^{m-1} \left(RB^j\delta, B^j\delta\right) - \left(RB^{m-1}(B\delta - Cv), B^{m-1}(B\delta - Cv)\right) \right], \tag{2.87}$$

$$\delta^\intercal(B^\intercal Q + QB)\delta \leq k(-R\delta, \delta) - k_1\|\delta\|^2, \tag{2.88}$$

for all $\begin{pmatrix}\delta \\ v\end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}$, where $R = \frac{B+B^*}{2}$, B^* being the adjoint of B with respect to the defined inner product (note that $(R\delta, \delta) = (B\delta, \delta)$).
Assume moreover that Q satisfies

$$(C^\intercal(Q^\intercal + Q)(B\delta - Cv))^2 \leq k \left[-\sum_{j=0}^{m-1} \left(RB^j\delta, B^j\delta\right) - \left(RB^{m-1}(B\delta - Cv), B^{m-1}(B\delta - Cv)\right) \right]. \tag{2.89}$$

For $(y_0, z_0, \delta_0)^\intercal \in D(\mathcal{A}^{m+1})$, define

$$G_0(t; y, \delta) = \int_{\Gamma_0} \delta^\intercal Q\delta ds + \int_{\Gamma_0} \alpha^\intercal \delta y ds,$$

with $\alpha = Q^\intercal C + QC$, and the Lyapunov functional

$$L_0(t) = N^2 \sum_{j=0}^{m} E_j(t) + \sqrt{N} G_0(t) + F_0(t),$$

with $N > 0$. Then for N large enough, there exists $C > 0$ (depending on N) such that

$$\frac{d}{dt} L_0(t) \leq -CE_0(t). \tag{2.90}$$

Proof. Deriving G_0 we obtain

$$\frac{d}{dt} G_0 = \int_{\Gamma_0} (\delta_t{}^\intercal Q\delta + \delta^\intercal Q\delta_t + \alpha^\intercal \delta_t y + \alpha^\intercal \delta y_t) ds$$

As $\delta_t = B\delta - Cy_t$, we deduce that

$$\frac{d}{dt} G_0 = \int_{\Gamma_0} (\delta^\intercal(B^\intercal Q + QB)\delta + y_t(\alpha^\intercal - C^\intercal Q - C^\intercal Q^\intercal)\delta + \alpha^\intercal \delta_t y) ds$$

As $\alpha = Q^\intercal C + QC$, we get

$$\frac{d}{dt} G_0 = \int_{\Gamma_0} (\delta^\intercal(B^\intercal Q + QB)\delta + \alpha^\intercal \delta_t y) ds. \tag{2.91}$$

By (2.85), (2.91), and the expression of the derivative of F_0, we have

$$\begin{aligned} \frac{d}{dt} L_0(t) \quad = \quad & N^2 \int_{\Gamma_0} (B\delta, \delta) ds + N^2 \sum_{j=1}^{m} \int_{\Gamma_0} (B\partial_t^j \delta, \partial_t^j \delta) ds + \sqrt{N} \int_{\Gamma_0} (\delta^\intercal(B^\intercal Q + QB)\delta + \alpha^\intercal \delta_t y) ds \\ & -\frac{1}{2} \int_\Omega (|y_t|^2 + |\nabla y|^2) dx - \frac{1}{2} \int_{\Gamma_0} q.\nu |\nabla y|^2 ds + \int_{\Gamma_0} C^\intercal M\delta q.\nabla y ds + \frac{1}{2} \int_{\Gamma_1} q.\nu |\nabla y|^2 ds \\ & +\frac{1}{2} \int_{\Gamma_0} q.\nu |y_t|^2 ds + \frac{d-1}{2} \int_{\Gamma_0} C^\intercal M\delta y ds. \end{aligned}$$

To deal with the term $\int_{\Gamma_0} q.\nu|y_t|^2 ds$, we first compute y_t using the condition $\delta_t = B\delta - Cy_t$ on Γ_0, multiplying by $C^\intercal$ we obtain

$$y_t = (C^\intercal C)^{-1}C^\intercal B\delta - (C^\intercal C)^{-1}C^\intercal \delta_t \text{ on } \Gamma_0. \tag{2.92}$$

Using a trace theorem, equality (2.92) together with Young's and Poincaré's inequalities we deduce that for all $\varepsilon_1 > 0, \varepsilon_2 > 0, \varepsilon_3 > 0$ we have

$$\begin{aligned}
\int_{\Gamma_0} q.\nu|y_t|^2 ds &\leq \int_{\Gamma_0} q.\nu|(C^\intercal C)^{-1}C^\intercal B\delta - (C^\intercal C)^{-1}C^\intercal\delta_t|^2 ds \qquad (2.93)\\
&\leq \int_{\Gamma_0} 2R_0\left[(C^\intercal C)^{-2}(C^\intercal B\delta)^2 + (C^\intercal C)^{-2}(C^\intercal\delta_t)^2\right] ds,\\
\sqrt{N}\int_{\Gamma_0} \alpha^\intercal\delta_t y ds &\leq \frac{N\varepsilon_1}{2}\int_{\Gamma_0}(\alpha^\intercal\delta_t)^2 ds + \frac{C_\gamma}{2\varepsilon_1}\int_\Omega |\nabla y|^2 dx, \qquad (2.94)\\
\int_{\Gamma_0} C^\intercal M\delta q.\nabla y ds &\leq \int_{\Gamma_0}\left[\frac{\varepsilon_2}{2}(C^\intercal M\delta)^2 + \frac{R_0^2}{2a\varepsilon_2}q.\nu|\nabla y|^2\right] ds, \qquad (2.95)\\
\frac{d-1}{2}\int_{\Gamma_0} C^\intercal M\delta y ds &\leq \frac{d-1}{2}\frac{\varepsilon_3}{2}\int_{\Gamma_0}(C^\intercal M\delta)^2 ds + \frac{d-1}{2}\frac{C_\gamma}{2\varepsilon_3}\int_\Omega |\nabla y|^2 dx. \qquad (2.96)
\end{aligned}$$

Choosing $\varepsilon_1, \varepsilon_2, \varepsilon_3$ so that we get $\frac{1}{8}\int_\Omega |\nabla y|^2 dx$, $\frac{1}{4}\int_{\Gamma_0} q.\nu|\nabla y|^2 ds$, and $\frac{1}{8}\int_\Omega |\nabla y|^2 dx$ on the right hand side of the inequalities (2.94), (2.95) and (2.96) respectively, and as $\frac{1}{2}\int_{\Gamma_1} q.\nu|y_t|^2 ds \leq 0$ we thus deduce that

$$\begin{aligned}
\frac{d}{dt}L_0(t) \leq\; & N^2\int_{\Gamma_0}(R\delta,\delta)ds + N^2\sum_{j=1}^m\int_{\Gamma_0}(R\partial_t^j\delta,\partial_t^j\delta)ds + K_1\int_{\Gamma_0}N(\alpha^\intercal\delta_t)^2 ds\\
&+\sqrt{N}\int_{\Gamma_0}(\delta^\intercal(B^\intercal Q + QB)\delta)ds - \frac{1}{4}\int_\Omega(|y_t|^2 + |\nabla y|^2)dx\\
&+R_0(C^\intercal C)^{-2}\int_{\Gamma_0}(C^\intercal\delta_t)^2 ds + R_0(C^\intercal C)^{-2}\int_{\Gamma_0}(C^\intercal B\delta)^2 ds + K_2\int_{\Gamma_0}(C^\intercal M\delta)^2 ds,
\end{aligned}$$

where K_1, K_2 are positive constants. But $\partial_t^j\delta = B^j\delta - \sum_{i=1}^j \partial_t^i y B^{j-i}C$, thus for all $1 \leq j \leq m-1$ and for all $1 \leq i \leq j$ we have $0 \leq j - i \leq j - 1 \leq m - 2$ and thus $RB^{j-i}C = 0$, and for all $2 \leq i \leq m$ we get $RB^{m-i}C = 0$ as $m - i \leq m - 2$. We deduce that for all $1 \leq j \leq m-1$, $(R\partial_t^j\delta, \partial_t^j\delta) = (RB^j\delta, B^j\delta)$ and $(R\partial_t^m\delta, \partial_t^m\delta) = (RB^m\delta - RB^{m-1}Cy_t, B^m\delta - B^{m-1}Cy_t)$. Hence, due to assumptions (2.87),(2.88), (2.89) we have,

$$\begin{aligned}
(C^\intercal\delta_t)^2 &\leq k\left(\sum_{j=1}^m(-R\partial_t^j\delta,\partial_t^j\delta) + \|\delta\|^2\right), \qquad (2.97)\\
\delta^\intercal(B^\intercal Q + QB)\delta &\leq k(-R\delta,\delta) - k_1\|\delta\|^2, \qquad (2.98)\\
(C^\intercal(Q^\intercal + Q)\delta_t)^2 &\leq -k\sum_{j=0}^m(R\partial_t^j\delta,\partial_t^j\delta). \qquad (2.99)
\end{aligned}$$

Clearly, we have $(C^\intercal B\delta)^2 + (C^\intercal M\delta)^2 \lesssim \|\delta\|^2$. Choosing N large enough we deduce that (2.90) holds. ■

Remark 2.8.4 *Substitute $\delta = 0_{\mathbb{R}^n}, v = 1$ in (2.87) we deduce that $(-RB^{m-1}C, B^{m-1}C) > 0$. Moreover, (2.88) implies that $0 \notin \sigma(B)$, otherwise there would exist $\delta \neq 0$ such that $B\delta = 0$ and thus (2.88) gives*

$$0 \leq -k_1\|\delta\|^2 < 0,$$

which is a contradiction.

Remark 2.8.5 *Define*

$$F_{1,k} = -\begin{pmatrix} B^\intercal CC^\intercal B & -B^\intercal CC^\intercal C \\ -C^\intercal CC^\intercal B & (C^\intercal C)^2 \end{pmatrix} + k\begin{pmatrix} M - \sum_{j=1}^{m}(B^j)^\intercal MRB^j & (B^m)^\intercal MRC_m \\ C_m^\intercal MRB^m & -C_m^\intercal MRC_m \end{pmatrix},$$

$$F_k = -\begin{pmatrix} B^\intercal \alpha\alpha^\intercal B & -B^\intercal \alpha\alpha^\intercal C \\ -C^\intercal \alpha\alpha^\intercal B & (C^\intercal \alpha)^2 \end{pmatrix} + k\begin{pmatrix} -\sum_{j=0}^{m}(B^j)^\intercal MRB^j & (B^m)^\intercal MRC_m \\ C_m^\intercal MRB^m & -C_m^\intercal MRC_m \end{pmatrix},$$

$$F_2^{k,k_1} = -kMR - k_1M - (B^\intercal Q + QB),$$

where $C_m = B^{m-1}C$. The assumptions of Proposition 2.8.3 can be restated as: there exist $k > 0, k_1 > 0$ and Q such that $F_k, F_{1,k}, F_2^{k,k_1}$ are positive definite matrices.

Proposition 2.8.6 *For N large enough, there exist $C_0 > 0$ and $C_1 > 0$ such that*

$$C_0\sum_{j=0}^{m} E_j(t) \leq L_0(t) \leq C_1\sum_{j=0}^{m} E_j(t). \tag{2.100}$$

Proof. With the convention, $\sum_{j=2}^{m} E_j(t) = 0$ if $m = 1$, we have

$$\begin{aligned} L_0(t) - N^2\sum_{j=2}^{m}E_j(t) &= N^2\int_\Omega(|\nabla y|^2 + |y_t|^2 + |\nabla y_t|^2 + |y_{tt}|^2)dx + N^2\int_{\Gamma_0}(\|\delta\|^2 + \|\delta_t\|^2)ds \\ &\quad + N^{\frac{1}{2}}\int_{\Gamma_0}(\delta^\intercal Q\delta + \alpha^\intercal \delta y)ds + \int_\Omega(y_t q.\nabla y + \frac{d-1}{2}yy_t)dx. \end{aligned}$$

Hence, by Cauchy-Schwarz's and Poincaré's inequalities and a trace theorem, we get

$$\begin{aligned} L_0(t) - N^2\sum_{j=2}^{m}E_j(t) &\leq N^2\left(\int_\Omega(|\nabla y|^2 + |y_t|^2)dx + \int_{\Gamma_0}\|\delta\|^2ds + \int_\Omega(|\nabla y_t|^2 + |y_{tt}|^2)dx + \int_{\Gamma_0}\|\delta_t\|^2ds\right) \\ &\quad + N^{\frac{1}{2}}K_3\int_{\Gamma_0} n\max_{1\leq i,j\leq n}|q_{i,j}|\|\delta\|^2ds + N^{\frac{1}{2}}\int_{\Gamma_0}\left(\frac{K_3|\alpha|^2\|\delta\|^2 + |y|^2}{2}\right)ds \\ &\quad + \frac{d-1}{2}\int_\Omega\left(\frac{|y_t|^2 + |y|^2}{2}\right)dx + \int_\Omega\left(|y_t|^2 + R_0^2|\nabla y|^2\right)dx. \end{aligned}$$

$$\begin{aligned}
L_0(t) - N^2\sum_{j=2}^{m} E_j(t) \quad &\geq \quad N^2\left(\int_\Omega (|\nabla y|^2 + |y_t|^2)dx + \int_{\Gamma_0} \|\delta\|^2 ds + \int_\Omega (|\nabla y_t|^2 + |y_{tt}|^2)dx + \int_{\Gamma_0} \|\delta_t\|^2 ds\right) \\
&\quad -N^{\frac{1}{2}}\int_{\Gamma_0} nK_3 \max_{1\leq i,j\leq n} |q_{i,j}| \|\delta\|^2 ds - N^{\frac{1}{2}} K_3 \frac{|\alpha|^2}{2}\int_{\Gamma_0} \|\delta\|^2\, ds - N^{\frac{1}{2}}\frac{C_\gamma}{2}\int_\Omega |\nabla y|^2\, dx \\
&\quad -\frac{d-1}{2}\int_\Omega \left(\frac{|y_t|^2 + C_\gamma |\nabla y|^2}{2}\right) dx - \int_\Omega \left(|y_t|^2 + R_0^2 |\nabla y|^2\right) dx,
\end{aligned}$$

where K_3 is a positive constant independent of N. For N chosen large enough, this right-hand side dominates $E_0(t) + E_1(t)$, and the conclusion follows. ■

Theorem 2.8.7 *Let $U_0 \in D(\mathcal{A}^m)$. Under the assumptions of Proposition 2.8.3, we get*

$$E_0(t) \lesssim \frac{1}{t}\sum_{j=0}^{m} E_j(0). \tag{2.101}$$

Proof. Let $U_0 \in D(\mathcal{A}^{m+1})$. Integrating the inequality (2.90) between 0 and $t > 0$, we get

$$\begin{aligned}
\int_0^t E_0(\tau)d\tau \quad &\leq \quad -\int_0^t C^{-1}\frac{d}{d\tau}L_0(\tau)d\tau = C^{-1}(L_0(0) - L_0(t)) \\
&\leq \quad C^{-1}C_1\sum_{j=0}^{m} E_j(0),
\end{aligned}$$

this last estimate following from (2.100). But $\frac{d}{dt}(tE_0(t)) = E_0(t) + t\frac{d}{dt}E_0(t) \leq E_0(t)$ and therefore

$$tE_0(t) = \int_0^t \frac{d(\tau E_0(\tau))}{d\tau}d\tau \leq \int_0^t E_0(\tau)d\tau \leq C^{-1}C_1\sum_{j=0}^{m} E_j(0),$$

as required.

By a density argument we deduce that (2.101) holds for all $U_0 \in D(\mathcal{A}^m)$. ■

Corollary 2.8.8 *Let $U_0 \in D(\mathcal{A})$. Under the assumptions of Proposition 2.8.3, we have*

$$\|e^{\mathcal{A}t}U_0\|_{\mathcal{H}}^2 \lesssim \frac{1}{t^{\frac{1}{m}}}\|U_0\|_{D(\mathcal{A})}^2. \tag{2.102}$$

Proof. Due to Theorem 2.8.7 we have $E(t) \leq \frac{C}{t}\sum_{j=0}^{m} E_j(0)$. By Remark 2.2.2 and Remark 2.8.4, $\mathcal{A}$ has a bounded inverse, thus applying Proposition 2.8.1 the result follows. ■

Proposition 2.8.9 *Consider the system (2.2) satisfying the assumptions of Proposition 2.8.3. For all $U_0 \in \mathcal{H}$, the energy of the solution of the system (2.2) decays asymptotically to zero, i.e.,*

$$E(t) \to 0, \;\; \textit{as} \quad t \to \infty.$$

Proof. Let $\varepsilon > 0$ be given. Due to the density of $D(\mathcal{A})$ in $\mathcal{H}$, there exists $U_0^\varepsilon \in D(\mathcal{A})$ such that $\|U_0^\varepsilon - U_0\| < \frac{\varepsilon}{2}$. But $(e^{\mathcal{A}t})_{t>0}$ is a contraction semigroup on $\mathcal{H}$, so $\|e^{\mathcal{A}t}(U_0^\varepsilon - U_0)\| < \frac{\varepsilon}{2}$. Since Corollary 2.8.8 yields that $e^{\mathcal{A}t}U_0^\varepsilon$ converges to zero as t tends to infinity, there exists $T_\varepsilon > 0$ such that for all $t > T_\varepsilon$, we have $\|e^{\mathcal{A}t}U_0^\varepsilon\|_{\mathcal{H}} < \frac{\varepsilon}{2}$ and hence $\|e^{\mathcal{A}t}U_0\|_{\mathcal{H}} < \frac{\varepsilon}{2} + \|e^{\mathcal{A}t}U_0^\varepsilon\|_{\mathcal{H}} < \varepsilon$, for all $t > T_\varepsilon$. ■

Remark 2.8.10 *If we suppose that $B \in C^{0,1}(\Gamma_0, M_n(\mathbb{R}))$ and $C \in C^{0,1}(\Gamma_0, \mathbb{R}^n)$ with the assumptions of Proposition 2.8.3 assumed to be valid for all $x \in \Gamma_0$ (with constants $k > 0, k_1 > 0$ independent of x) with $Q \in C^{0,1}(\Gamma_0, M_n(\mathbb{R}))$, then the results of Propositions 2.8.3, 2.8.7, and 2.8.9 are still valid (proved with the same arguments).*

2.9 Examples

In this section, we illustrate our general framework by checking the assumptions of the frequency domain approach and that of the multiplier method for some particular examples.

2.9.1 Example 1: Acoustic boundary conditions

Let Ω be a domain of $\mathbb{R}^3$ with a boundary Γ divided as in the introduction. The system considered by Beale [16] (with $\Gamma_1 = \emptyset$) and Rivera-Qin in [38] with $c = 1$ is the following one:

$$\begin{cases} y_{tt}(x,t) - \Delta y(x,t) = 0 & , x \in \Omega, t > 0, \\ y(x,t) = 0 & , x \in \Gamma_1, t > 0, \\ \frac{\partial y}{\partial \nu}(x,t) = \eta_t(x,t) & , x \in \Gamma_0, t > 0, \\ m\eta_{tt}(x,t) + d\eta_t(x,t) + k\eta(x,t) + \rho y_t(x,t) = 0, & , x \in \Gamma_0, t > 0, \end{cases}$$

where ρ is a positive constant and m, d, k are positive and sufficiently smooth functions on Γ_0. Let us set $m_0 = \min\limits_{x\in\Gamma_0} m(x), m_1 = \max\limits_{x\in\Gamma_0} m(x)$, $d_0 = \min\limits_{x\in\Gamma_0} d(x)$ and $d_1 = \max\limits_{x\in\Gamma_0} d(x)$.

We readily check that this system can be rewritten in the form of system (2.2) with $\delta = (\eta, \eta_t)^\top$ and

$$B(x) = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{pmatrix}, M(x) = \begin{pmatrix} \frac{k}{\rho} & 0 \\ 0 & \frac{m}{\rho} \end{pmatrix}, R(x) = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{d}{m} \end{pmatrix}, C(x) = \begin{pmatrix} 0 \\ \frac{\rho}{m} \end{pmatrix}, x \in \Gamma_0.$$

For all $x \in \Gamma_0$, the matrix $B(x)$ is Hurwitz and thus $\Sigma \cap i\mathbb{R} = \emptyset$. Hence, the assumptions $(A_2), (A_3)$ and (A_4) of Proposition 2.4.2 hold. Moreover, we can easily check (A_1) and we then deduce by the proof of Proposition 2.4.3 that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$ if Γ_1 is non empty. In addition, the inequality (2.57) of Proposition 2.6.2 is satisfied for $p = 1$. Therefore, if Γ_1 is non empty and if e^{At} is exponentially stable (see Remark 2.6.5), we deduce by Proposition 2.6.2 that

$$E(t) \lesssim \frac{1}{t^{2/3}} \|U_0\|^2_{D(\mathcal{A})}, \ \forall t > 0.$$

A similar result holds if e^{At} is only polynomially stable, in particular in the setting of Remark 2.6.6, we will get

$$E(t) \lesssim \frac{1}{t^{\frac{2}{5}}} \|U_0\|^2_{D(\mathcal{A})} \ \forall t > 0.$$

On the other hand if Γ_1 is empty, since B is invertible and $(B^{-1}C, C) = 0$, we are then in the second case of section 2.7 (subsection 2.7.2) as we easily check that the assumptions (2.79) and (2.80) are satisfied. Therefore, if e^{At} is exponentially stable (see Remark 2.6.5), we deduce by Proposition 2.7.15 that

$$\|(y, y_t, \eta, \eta_t)^\top - \alpha(1,0,0,0)^\top - \beta(t, 1, -\frac{\rho}{k}, 0)^\top\|_{\widetilde{\mathcal{H}}} \lesssim \frac{1}{t^{2/3}} \|U_0\|^2_{D(\mathcal{A})}, \ \forall t > 0,$$

where α and β are two constants depending on U_0.

If we want to apply the multiplier method of section 2.8, we set

$$Q = \begin{pmatrix} \frac{d}{2} & \frac{m}{2} \\ \frac{m}{2} & 0 \end{pmatrix}, \alpha = \begin{pmatrix} \rho \\ 0 \end{pmatrix},$$

and directly obtain that

$$B^\intercal Q + QB = \begin{pmatrix} -k & 0 \\ 0 & m \end{pmatrix}, B\delta - Cv = \begin{pmatrix} \delta_2 \\ -\frac{k}{m}\delta_1 - \frac{d}{m}\delta_2 - \frac{\rho}{m}v \end{pmatrix},$$

for $\delta = (\delta_1, \delta_2)^\intercal \in \mathbb{R}^2$ and $v \in \mathbb{R}$. It is easy to see that

$$\begin{aligned} (C^\intercal(B\delta - Cv))^2 &= \frac{\rho^2}{m^2}\left(-\frac{k}{m}\delta_1 - \frac{d}{m}\delta_2 - \frac{\rho}{m}v\right)^2 \leq \frac{\rho^2}{m_0^2}\frac{m_1}{d_0}\frac{d_0}{m_1}\left(-\frac{k}{m}\delta_1 - \frac{d}{m}\delta_2 - \frac{\rho}{m}v\right)^2 \\ &\lesssim \frac{d}{m}\left(-\frac{k}{m}\delta_1 - \frac{d}{m}\delta_2 - \frac{\rho}{m}v\right)^2 = -\left(R(B\delta - Cv), (B\delta - Cv)\right), \end{aligned}$$

$$(\alpha^\intercal(B\delta - Cv))^2 = \rho^2\delta_2^2 \quad \leq \quad \rho^2\frac{m_1}{d_0}(-R\delta, \delta),$$

that are satisfied for all $x \in \Gamma_0$. Hence, the assumptions of Proposition 2.8.3 holds with $m = 1$ independently of $x \in \Gamma_0$, thus according to Remark 2.8.10 the energy of the system decays polynomially for initial datum $U_0 \in D(\mathcal{A})$ and we thus get the decay rate (2.101) with $m = 1$ as in [38].

2.9.2 Example 2

Consider the following system

$$\begin{cases} y_{tt}(x,t) - \Delta y(x,t) = 0 & , x \in \Omega, t > 0, \\ y(x,t) = 0 & , x \in \Gamma_1, t > 0, \\ \frac{\partial y}{\partial \nu}(x,t) = b_1\delta(t) + \delta_t(t) - \kappa(t) & , x \in \Gamma_0, t > 0, \\ \delta_{tt}(x,t) + b_1\delta_t(x,t) + b_0\delta(x,t) + b_0 y_t(x,t) = 0 & , x \in \Gamma_0, t > 0, \\ \kappa_t(t) + b_2\kappa(t) - y_t(x,t) = 0 & , x \in \Gamma_0, t > 0, \end{cases} \tag{2.103}$$

with b_0, b_1, b_2 positive constants. Choosing

$$\delta_1 = \frac{\delta_t + b_1\delta}{b_0}, \delta_2 = -\delta, \delta_3 = -\kappa,$$

we get a system of the form (2.2) with

$$M = \begin{pmatrix} b_0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 \\ -b_0 & -b_1 & 0 \\ 0 & 0 & -b_2 \end{pmatrix}, R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -b_1 & 0 \\ 0 & 0 & -b_2 \end{pmatrix}, C = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

As in the preceding example, if Γ_1 is non empty, it is easy to check that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$ ((A_1) holds and $\Sigma \cap i\mathbb{R} = \emptyset$) as well as (2.57) with $p = 1$ (by Proposition 2.6.4 as $P(C) \neq 0$). Hence if Γ_1 is non empty and if e^{At} is exponentially stable, the energy of the solution of (2.103) satisfies

$$E(t) \lesssim \frac{1}{t^{2/3}} \|U_0\|^2_{D(\mathcal{A})},\ \forall U_0 \in D(\mathcal{A}),\ \forall t > 0.$$

For this example, if Γ_1 is empty, B is also invertible but $(B^{-1}C, C) \neq 0$. Therefore, we are in the first case of section 2.7 (subsection 2.7.1). Hence, if e^{At} is exponentially stable (see Remark 2.6.5), we deduce by Proposition 2.7.15 that

$$\|(y - \alpha, y_t, \delta_1, \delta_2, \delta_3)^\top\|_{\mathcal{H}} \lesssim \frac{1}{t^{2/3}} \|U_0\|^2_{D(\mathcal{A})},\ \forall t > 0,$$

with α a constant depending on U_0.

Let us now check the assumptions of Proposition 2.8.3 with $m = 1$ and the choice

$$Q = \begin{pmatrix} \frac{b_1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For any $\delta = (\delta_1, \delta_2, \delta_3)^\intercal \in \mathbb{R}^3$ and $v \in \mathbb{R}$, simple calculations yield

$$B^\intercal Q + QB = \begin{pmatrix} -b_0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B\delta - Cv = \begin{pmatrix} \delta_2 - v \\ -b_0\delta_1 - b_1\delta_2 \\ -b_2\delta_3 - v \end{pmatrix}, \alpha = \begin{pmatrix} b_1 \\ 1 \\ 0 \end{pmatrix}$$

and

$$-(R(B\delta - Cv), (B\delta - Cv)) = b_1(b_0\delta_1 + b_1\delta_2)^2 + b_2(b_2\delta_3 + v)^2.$$

Moreover, for any $\delta = (\delta_1, \delta_2, \delta_3)^\intercal \in \mathbb{R}^3$ and $v \in \mathbb{R}$, we directly check that

$$\begin{aligned} (C^\intercal(B\delta - Cv))^2 &= (\delta_2 - 2(b_2\delta_3 + v) + b_2\delta_3)^2 \\ &\lesssim \delta_2^2 + \delta_3^2 + (b_2\delta_3 + v)^2, \end{aligned}$$

and

$$\begin{aligned} (\alpha^\intercal(B\delta - Cv))^2 &= (b_1\delta_2 - b_1(v + b_2\delta_3) + b_1b_2\delta_3 - b_0\delta_1 - b_1\delta_2)^2 \\ &\lesssim b_1^2\delta_2^2 + (b_2b_1)^2\delta_3^2 + b_1^2(v + b_2\delta_3)^2 + (b_0\delta_1 + b_1\delta_2)^2. \end{aligned}$$

Thus, we have

$$(\alpha^\intercal(B\delta - Cv))^2 + (C^\intercal(B\delta - Cv))^2 \lesssim (-R\delta, \delta) - (R(B\delta - Cv), (B\delta - Cv)).$$

This implies that (2.87) and (2.89) hold. Finally, due to the definition of R and the form of $B^\intercal Q + QB$, (2.88) is valid with $k_1 = b_0$ and k large enough. Hence, the polynomial stability follows from Theorem 2.8.7 and the energy of the system (2.103) defined on a domain whose boundary is divided as in section 2.8 satisfies (2.101) with $m = 1$.

2.9.3 Example 3

Consider the following system

$$\begin{cases} y_{tt}(x,t) - \Delta y(x,t) = 0 & , x \in \Omega, t > 0, \\ y(x,t) = 0 & , x \in \Gamma_1, t > 0, \\ \frac{\partial y}{\partial \nu}(x,t) - b_1 \kappa(x,t) - \kappa_t(x,t) = 0 & , x \in \Gamma_0, t > 0, \\ \kappa_{tt}(t) + b_1 \kappa_t(t) + b_0 \kappa(t) + b_0 y_t(x,t) = 0 & , x \in \Gamma_0, t > 0, \end{cases} \tag{2.104}$$

Set $\delta = (\frac{b_1 \kappa + \kappa_t}{b_0}, -\kappa)^\intercal = (\delta_1, \delta_2)^\intercal$, then our system is nothing but (2.2) with

$$M = \begin{pmatrix} b_0 & 0 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -b_0 & -b_1 \end{pmatrix}, R = \begin{pmatrix} 0 & 0 \\ 0 & -b_1 \end{pmatrix}, C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In this example, if Γ_1 is non empty, we have $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$ and (2.57) holds for $p = 2$ ($P(B_0 C) \neq 0$, see Proposition 2.6.4). Hence, if Γ_1 is non empty and if e^{At} is exponentially stable, then by Proposition 2.6.2 we obtain

$$E(t) \lesssim \frac{1}{t^{2/5}} \|U_0\|^2_{D(\mathcal{A})}, \ \forall t > 0.$$

If Γ_1 is empty, then we are in the first case of section 2.7 (subsection 2.7.1) as B is invertible and $(B^{-1}C, C) \neq 0$. Therefore, if e^{At} is exponentially stable, then we deduce by Proposition 2.7.15 that

$$\|(y - \alpha, y_t, \delta_1, \delta_2, \delta_3)^\top\|_{\mathcal{H}} \lesssim \frac{1}{t^{2/5}} \|U_0\|^2_{D(\mathcal{A})}, \ \forall t > 0,$$

with α a constant depending on U_0.

In the setting of section 2.8, we first note that (2.89) is not valid with $m = 1$ and thus the assumptions of Proposition 2.8.3 are not satisfied for $m = 1$. Let us nevertheless check the assumptions of Proposition 2.8.3 for $m = 2$. Indeed, choosing

$$Q = \begin{pmatrix} \frac{b_1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix},$$

we get

$$B^\intercal Q + QB = \begin{pmatrix} -b_0 & 0 \\ 0 & 1 \end{pmatrix}, \alpha = 2QC = \begin{pmatrix} b_1 \\ 1 \end{pmatrix}, B\delta - Cv = \begin{pmatrix} \delta_2 - v \\ -b_0\delta_1 - b_1\delta_2 \end{pmatrix}.$$

Since $RC = 0$, we have

$$-(R(B\delta - Cv), (B\delta - Cv)) = -(RB\delta, B\delta) = b_1(b_0\delta_1 + b_1\delta_2)^2,$$

and

$$-(RB(B\delta - Cv), B(B\delta - Cv)) = b_1 \left(b_0(\delta_2 - v) - b_1(b_0\delta_1 + b_1\delta_2)\right)^2.$$

Indeed,

$$\begin{aligned} (C^\intercal(B\delta - Cv))^2 &= \frac{(b_0(\delta_2 - v))^2}{b_0^2} = \frac{1}{b_0^2}(b_0(\delta_2 - v) - b_1(b_0\delta_1 + b_1\delta_2) + b_1(b_0\delta_1 + b_1\delta_2))^2 \\ &\leq -\frac{2}{b_1 b_0^2}\left(RB(B\delta - Cv), B(B\delta - Cv)\right) - \frac{2b_1}{b_0^2}\left(RB\delta, B\delta\right), \end{aligned}$$

and

$$\begin{aligned}(\alpha^{\intercal}(B\delta - Cv))^2 &= (b_1(\delta_2 - v) - b_0\delta_1 - b_1\delta_2)^2 \leq 2(b_1^2(\delta_2 - v)^2 + (b_0\delta_1 + b_1\delta_2)^2) \\ &\leq -\frac{4b_1}{b_0^2}\,(RB(B\delta - Cv), B(B\delta - Cv)) - (\frac{4b_1^3}{b_0^2} + \frac{2}{b_1})\,(RB\delta, B\delta)\,.\end{aligned}$$

Hence, the assumptions of Proposition 2.8.3 holds with $m = 2$. According to Corollary 2.8.8, the energy of system (2.104) therefore satisfies the following polynomial estimate

$$E(t) \lesssim \frac{1}{t^{\frac{1}{2}}}\|U_0\|^2_{D(\mathcal{A})}.$$

3 Stabilization of second order evolution equations by unbounded dynamic feedbacks and applications

3.1 Introduction

Let X be a complex Hilbert space with norm and inner product denoted respectively by $\|\cdot\|_X$ and $<\cdot,\cdot>_{X,X}$. Let A be a linear unbounded positive self-adjoint operator which is the Friedrichs extension of the triple (X, V, a), where a is a closed quadratic form with domain V dense in X. Note that by definition $\mathcal{D}(A)$ (the domain of A) is dense in X and $\mathcal{D}(A)$ equipped with the graph norm is a Hilbert space and the embedding $\mathcal{D}(A) \subset X$ is continuous. Further, let U be a complex Hilbert space (which will be identified with its dual space) with norm and inner product respectively denoted by $\|\cdot\|_U$ and $<\cdot,\cdot>_{U,U}$, let $\widehat{C}$ be a Linear operator on U and let $B \in \mathcal{L}(U, V')$, where V' is the dual space of V obtained by means of the inner product in X. Denote by $B^* \in \mathcal{L}(V, U)$ the adjoint of B. Consider the system

$$\begin{cases} x''(t) + Ax(t) + Bu(t) = 0, & t \in [0, +\infty) \\ \rho u'(t) - \widehat{C}u(t) - B^*x'(t) = 0, & t \in [0, +\infty) \\ x(0) = x_0, x'(0) = y_0, u(0) = u_0, \end{cases} \tag{3.1}$$

with ρ a scalar parameter. By replacing ρ by 0 and $-\widehat{C}$ by the identity in system (3.1) we obtain the system whose stability was studied in [9].

In this chapter we are interested in studying the stability of linear control problems coming from elasticity which can be written as

$$\begin{cases} x''(t) + Ax(t) + Bu(t) = 0, & t \in [0, +\infty) \\ u'(t) - \widehat{C}u(t) - B^*x'(t) = 0, & t \in [0, +\infty) \\ x(0) = x_0, x'(0) = y_0, u(0) = u_0, \end{cases} \tag{3.2}$$

where $x : [0, +\infty) \to X$ is the state of the system, $u \in L^2(0, T; U)$ is the input function and $\widehat{C}$ is a m-dissipative operator on U. We denote the differentiation with respect to time by $'$.

The aim of this chapter is to give sufficient conditions leading to the uniform or non uniform stability of the solutions of the corresponding closed loop system.

The second equation of the considered system describes a dynamical control in some models. Some systems that can be covered by the formulation (3.2) are for example the hybrid systems.

This chapter is organized as follows. In the next section we justify the well-posedness of the problem then we write $\widehat{C}$ as a sum of a skew-adjoint operator $-C$ and a self-adjoint operator $-DD^*$. The case where the operator D is bounded is studied in section 3.3. Under a regularity assumption we prove in section 3.4 that the observability properties of the undamped problem, obtained by replacing $\widehat{C}$ in system (3.2) by $-C$, imply uniform decay estimates for the damped problem (3.2). In section 3.5 we state the results concerning the polynomial stability of the energy. Finally, we present in section 3.6 some examples as applications of the general setting where we obtain using a variety of methods polynomial or exponential energy decay rates.

3.2 Well-posedness results

In order to study the system (3.2) we use a reduction order argument. First, we introduce the Hilbert space $\mathcal{H} = V \times X \times U$ equipped with the scalar product

$$< z, \tilde{z} >_{\mathcal{H},\mathcal{H}} = a(x, \tilde{x}) + < y, \tilde{y} >_{X,X} + < u, \tilde{u} >_{U,U}, \quad \forall z, \tilde{z} \in \mathcal{H}, z = (x, y, u), \tilde{z} = (\tilde{x}, \tilde{y}, \tilde{u}).$$

Then we consider the unbounded dissipative operator, see Proposition 3.2.1 , denoted by A_d

$$\begin{aligned} \mathcal{A}_d : \ & \mathcal{D}(\mathcal{A}_d) \longrightarrow \mathcal{H} \\ & z = (x, y, u) \longmapsto \mathcal{A}_d z = (y, -Ax - Bu, B^* y + \widehat{C} u), \end{aligned}$$

where

$$\mathcal{D}(\mathcal{A}_d) = \{(x, y, u) \in V \times V \times D(\widehat{C}), Ax + Bu \in X\}.$$

So the system (3.2) is formally equivalent to

$$z'(t) = \mathcal{A}_d z(t), z(0) = z_0, \tag{3.3}$$

where $z_0 = (x_0, y_0, u_0)$.

Proposition 3.2.1 *The operator $\mathcal{A}_d$ is an m-dissipative operator on $\mathcal{H}$ and thus it generates a C_0-semigroup.*

Proof.

$$\begin{aligned} < \mathcal{A}_d z, z >_{\mathcal{H},\mathcal{H}} &= a(y, x) - < Ax + Bu, y >_{X,X} + < B^* y + \widehat{C} u, u >_{U,U} \\ &= a(y, x) - a(x, y) - < Bu, y >_{V',V} + < B^* y, u >_{U,U} + < \widehat{C} u, u >_{U,U} \\ &= a(y, x) - a(x, y) + < \widehat{C} u, u >_{U,U} . \end{aligned}$$

Taking the real part of the above identity we get (3.5) since $\widehat{C}$ is dissipative. Hence $\mathcal{A}_d$ is dissipative.
We would like to show that there exists $\lambda > 0$ such that $\lambda I - \mathcal{A}_d$ is surjective. Let $\lambda > 0$ be given. Clearly, we have $\lambda \notin \sigma(\widehat{C})$. For $(f, g, h) \in \mathcal{H}$, we look for $(x, y, u) \in \mathcal{D}(\mathcal{A}_d)$ such that

$$(\lambda I - \mathcal{A}_d) \begin{pmatrix} x \\ y \\ u \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix},$$

i.e. we are searching for $x \in V, \ y \in V, \ u \in D(\widehat{C})$ satisfying

$$\begin{aligned} \lambda x - y &= f \\ \lambda^2 x + Ax + Bu &= g + \lambda f \\ (\lambda I - \widehat{C}) u - B^* y &= h. \end{aligned}$$

By Lax-Milgram lemma there exists a unique $x \in V$ such that

$$\left(\lambda^2 + A + \lambda B(\lambda I - \widehat{C})^{-1}B^*\right) x = g + \lambda f + B(\lambda I - \widehat{C})^{-1} \left(B^* f - h\right).$$

In fact, we have $\lambda^2 + A + \lambda B(\lambda I - \widehat{C})^{-1}B^* \in \mathcal{L}(V, V')$, $g + \lambda f + B(\lambda I - \widehat{C})^{-1} \left(B^* f - h\right) \in V'$ and

$$\Re \left\langle \left(\lambda^2 + A + \lambda B(\lambda I - \widehat{C})^{-1}B^*\right) x, x \right\rangle_{V',V} \geq \langle Ax, x \rangle_{V',V},$$

since

$$\begin{aligned} \Re \left\langle B(\lambda I - \widehat{C})^{-1}B^* x, x \right\rangle_{V',V} &= \Re \left\langle u, (\lambda I - \widehat{C})u \right\rangle_{U,U} \\ &= \lambda \|u\|^2 - \Re \left\langle u, \widehat{C}u \right\rangle_{U,U} \geq 0, \end{aligned}$$

with $u = (\lambda I - \widehat{C})^{-1}B^* x$, i.e. the coercivity property is satisfied.

Define

$$u = (\lambda I - \widehat{C})^{-1} \left(h + B^* \left(\lambda x - f\right)\right),$$

by choosing $y = \lambda x - f$ we deduce the surjectivity of $\lambda I - \mathcal{A}$. Finally, we conclude that $\lambda I - \mathcal{A}$ is bijective, for all $\lambda > 0$.

■

Now, we are able to state the following existence result of problem (3.3).

Proposition 3.2.2 *(i) For an initial datum $z_0 \in \mathcal{H}$, there exists a unique solution $z \in C([0, +\infty), \mathcal{H})$ to system (3.3). Moreover, if $z_0 \in \mathcal{D}(\mathcal{A}_d)$, then*

$$z \in C([0, +\infty), \mathcal{D}(\mathcal{A}_d)) \cap C^1([0, +\infty), \mathcal{H}). \tag{3.4}$$

(ii) For each $z_0 \in \mathcal{D}(\mathcal{A}_d)$, the energy $E(t)$ of the solution z of (3.3), defined by

$$E(t) = \frac{1}{2}\|z(t)\|^2_{\mathcal{H}},$$

satisfies

$$E'(t) = \Re < \widehat{C}u(t), u(t) > \leq 0, \tag{3.5}$$

therefore the energy is non-increasing.

Moreover, we have the following estimate

$$-\int_0^t \Re < \widehat{C}u(s), u(s) > ds = E(0) - E(t) \leq \frac{1}{2}\|z_0\|^2_{\mathcal{H}}, \ \forall t \in [0, +\infty), \forall z_0 \in \mathcal{H}. \tag{3.6}$$

Proof. (i) is a direct consequence of Lumer-Phillips theorem (see [42]).

(ii) For an initial datum in $\mathcal{D}(\mathcal{A}_d)$ from (3.4), we know that u is of class C^1 in time, thus we can derive the energy $E(t)$, and using Propostion 3.2.1 we obtain:

$$E'(t) = \Re < z', z >_{\mathcal{H},\mathcal{H}} = \Re < \mathcal{A}_d z, z >_{\mathcal{H},\mathcal{H}} = \Re < \widehat{C}u, u > .$$

Hence the energy is non-increasing. Finally (3.6) is a direct consequence of (3.5). ■

Assume that $\widehat{C}$ can be written as $\widehat{C} = -C - DD^*$ where C is a skew-adjoint operator on U, $D \in \mathcal{L}(W, (D(C))')$, and W is supposed to be a Hilbert subspace of U identified with its dual, thus $D^* \in \mathcal{L}(D(C), W)$.

We first introduce the conservative system associated with the initial problem (3.2) as

$$\begin{cases} x_1''(t) + Ax_1(t) + Bu_1(t) = 0, & t \in (0, +\infty) \\ u_1'(t) + Cu_1(t) - B^*x_1'(t) = 0, & t \in (0, +\infty) \\ x_1(0) = x_0, x_1'(0) = y_0, u_1(0) = u_0. \end{cases} \tag{3.7}$$

Denote by $\mathcal{A}_c$ the operator obtained by replacing $\widehat{C}$ by $-C$ in the expression $\mathcal{A}_d$. Thus $\mathcal{A}_c$ is given by

$$\mathcal{A}_c z_1 = (y_1, -Ax_1 - Bu_1, B^*y_1 - Cu_1), \, \forall z_1 = (x_1, y_1, u_1) \in \mathcal{D}(\mathcal{A}_c),$$

with

$$\mathcal{D}(\mathcal{A}_c) = \{(x_1, y_1, u_1) \in V \times V \times D(C), Ax_1 + Bu_1 \in X\}.$$

The corresponding Cauchy problem can be written as

$$z_1'(t) = \mathcal{A}_c z_1(t), z_1(0) = z_0 \in \mathcal{D}(\mathcal{A}_c). \tag{3.8}$$

We can easily check that $\mathcal{A}_c$ is closed anti-symmetric, maximal dissipative operator whose opposite $-\mathcal{A}_c$ is also maximal dissipative, therefore $\mathcal{A}_c$ is skew-adjoint and generates a unitary group.

Denote by $\mathcal{A}_r$ the operator

$$\mathcal{A}_r : (x, y, u) \in \mathcal{H} \mapsto (0, 0, -DD^*u),$$

it is easy to see that $\mathcal{A}_r$ is dissipative and $\mathcal{A}_d = \mathcal{A}_c + \mathcal{A}_r$. Note that the energy satisfies:

$$E'(t) = -\|D^*u(t)\|_W^2. \tag{3.9}$$

3.3 Some regularity results

Let $T > 0$ be fixed and $u \in L^2(0, T; U)$ be the last component of the solution z of (3.3). Consider the evolution problem

$$z_2'(t) = \mathcal{A}_c z_2(t) + \mathcal{A}_r z(t), z_2(0) = 0, t \in [0, T], \tag{3.10}$$

where $\mathcal{A}_r z(t) = -(0, 0, DD^*u(t))$.

Lemma 3.3.1 *Suppose that $D \in \mathcal{L}(U)$. Then problem (3.10) admits a unique solution $z_2(t) = (x_2(t), y_2(t), u_2(t))$ such that*

$$u_2 \in L^2(0, T; U),$$

satisfying the following estimate

$$\|D^*u_2\|_{L^2(0,T;U)} \leq c\|D^*u\|_{L^2(0,T;U)}, \tag{3.11}$$

where c is a positive constant.

Proof. Clearly $\mathcal{A}_r z(t) = -(0,0,DD^*u(t)) \in C^1(0,T;\mathcal{H})$, and since $\mathcal{A}_c$ generates a unitary group $e^{\mathcal{A}_c \cdot}$ on $\mathcal{H}$, then (3.10) admits a unique solution given by

$$z_2(t) = \int_0^t e^{\mathcal{A}_c(t-s)}\mathcal{A}_r z(s)ds = \int_0^t e^{\mathcal{A}_c(s)}\mathcal{A}_r z(t-s)ds, \forall t \in [0,T].$$

Moreover $D \in \mathcal{L}(U)$ and

$$\begin{aligned}
\|u_2\|^2_{L^2(0,T;U)} &= \int_0^T \|u_2(t)\|^2_U dt \\
&\leq \int_0^T \|z_2(t)\|^2_{\mathcal{H}} dt \\
&\leq \int_0^T \| \int_0^t e^{\mathcal{A}_c s}\ \mathcal{A}_r z(t-s)ds\|^2_{\mathcal{H}} dt \\
&\leq \int_0^T \left(\int_0^t \|e^{\mathcal{A}_c s}\|\ \|\mathcal{A}_r z(t-s)\|_{\mathcal{H}} ds\right)^2 dt,
\end{aligned}$$

thus

$$\begin{aligned}
\|u_2\|^2_{L^2(0,T;U)} &\leq \int_0^T (\int_0^t \|\mathcal{A}_r z(s)\|_{\mathcal{H}} ds)^2 dt \\
&\leq \int_0^T (\int_0^T \|\mathcal{A}_r z(s)\|_{\mathcal{H}} ds)^2 dt \\
&\leq \int_0^T (\int_0^T \|DD^*u(s)\|_U ds)^2 dt \\
&\leq \int_0^T (\int_0^T 1^2 ds)(\int_0^T \|DD^*u(s)\|^2_U ds) dt \\
&\leq \int_0^T T\|D\|^2\|D^*u\|^2_{L^2(0,T;U)} dt \\
&\leq T^2\|D\|^2\|D^*u\|^2_{L^2(0,T;U)}.
\end{aligned}$$

Consequently, as $\|D^*u_2\|_{L^2(0,T;U)} \leq \|D^*\|\|u_2\|_{L^2(0,T;U)}$, (3.11) holds with the constant $T\|D\|\|D^*\|$. ■

3.4 Uniform stability

In this section, we give sufficient and necessary conditions which lead to uniform stability of system (3.3).

Recall that the conservative system (3.8) is obtained by replacing $\widehat{C}$ by $-C$ in system (3.3) and that Proposition 3.2.2 still holds. In order to get uniform stability we will need the following assumptions:

(**O**) (Observability inequality) There exists a time $T > 0$ and a constant $c(T) > 0$ (which only depends on T) such that, for all $z_0 \in \mathcal{D}(\mathcal{A}_c)$, the solution $z_1(t) = (x_1(t), y_1(t), u_1(t))$ of (3.8) satisfies the following observability estimate:

$$\int_0^T \|D^*u_1(s)\|^2_W ds \geq c(T)\|z_0\|^2_{\mathcal{H}}. \tag{3.12}$$

(**H**) (Transfer function estimate) Assume that for every $\lambda \in \mathbb{C}_+ = \{\lambda \in \mathbb{C} | \Re\lambda > 0\}$

$$\mathbb{C}_+ \ni \lambda \to H(\lambda) = -D^*(\lambda I + C + \lambda B^*(\lambda^2 + A)^{-1}B)^{-1}D \in \mathcal{L}(W),$$

is bounded on $C_\beta = \{\lambda \in \mathbb{C} | \Re\lambda = \beta\}$, where β is a positive constant.

Theorem 3.4.1 *Assume that assumption* (**H**) *is satisfied or that $D \in \mathcal{L}(U)$. Then system (3.3) is exponentially stable, which means that the energy of the system satisfies*

$$E(t) \leq c\, e^{-\omega t}\, E(0), \forall t \in [0, +\infty), \tag{3.13}$$

where c and ω are two positive constants independent of the initial data $z_0 \in \mathcal{D}(\mathcal{A}_d)$ if and only if the inequality (3.12) is satisfied.

By using [36, Theorem 5.1] and [7, Proposition 2.1] we have the following characterization of the uniform stabily of (3.3) by a frequency criteria (Hautus test).

Corollary 3.4.2 *Assume that assumption* (**H**) *is satisfied or that $D \in \mathcal{L}(U)$. Then system (3.3) is exponentially stable in the energy space if and only if there exists a constant $\delta > 0$ such that for all $w \in \mathbb{R}, z \in \mathcal{D}(\mathcal{A}_c)$ we have*

$$\|(iw - \mathcal{A}_c)z\|^2_{\mathcal{H}} + \left\| \begin{pmatrix} 0 & 0 & D^* \end{pmatrix} z \right\|^2_U \geq \delta \, \|z\|^2_{\mathcal{H}}. \tag{3.14}$$

Proof. (of Theorem 3.4.1). Let $z(t) = (x(t), y(t), u(t))$ be the solution of (3.3) with initial datum $z_0 \in \mathcal{D}(\mathcal{A}_d)$. Consider $z_1(t) = (x_1(t), y_1(t), u_1(t))$ the solution of (3.8) with initial datum $z_0 \in \mathcal{D}(\mathcal{A}_d)$. Let $z_2(t) = (x_2(t), y_2(t), u_2(t))$ be such that $z_2(t) = z(t) - z_1(t)$. Then z_2 is solution of (3.10) and due to Lemma 3.3.1 its last component u_2 satisfies (3.11) if $D \in \mathcal{L}(U)$. Otherwise, (3.11) holds true due to assumption (**H**). Since $u = u_1 + u_2$, we get

$$\begin{array}{lcll} \|z_0\|^2_{\mathcal{H}} & \lesssim & \|D^* u_1\|^2_{L^2(0,T;W)} & \text{estimate (3.12)} \\ & \lesssim & \|D^* u\|^2_{L^2(0,T;W)} + \|D^* u_2\|^2_{L^2(0,T;W)} & \text{(triangle inequality)} \\ & \lesssim & \|D^* u\|^2_{L^2(0,T;W)} & \text{(estimate (3.11)).} \end{array}$$

Indeed x_2, u_2 satisfies the system

$$\left\{ \begin{array}{ll} x_2''(t) + Ax_2(t) + Bu_2(t) = 0, & t \in (0, +\infty) \\ u_2'(t) + Cu_2(t) - B^* x_2'(t) = -DD^* u(t), & t \in (0, +\infty) \\ x_2(0) = 0, x_2'(0) = 0, u_2(0) = 0. & \end{array} \right. \tag{3.15}$$

Extend $D^* u$ by zero on $\mathbb{R} \setminus [0, T]$. Since the system (3.15) is reversible by time we solve the system on $\mathbb{R}$. We obtain a function $z \in C(\mathbb{R}; V) \cap C^1(\mathbb{R}; V) \cap L^2(\mathbb{R}; V)$ which is null for all $t \leq 0$.

Let $\widehat{x}_2(\lambda)$ and $\widehat{u}_2(\lambda)$, where $\lambda = \gamma + i\eta$, $\Re(\lambda) = \gamma > 0$ and $\eta \in \mathbb{R}$, be the respective Laplace transforms of x_2 and u_2 with respect to t. Then $\widehat{x}_2$ and $\widehat{u}_2$ satisfy

$$\left\{ \begin{array}{l} \lambda^2 \widehat{x}_2(\lambda) + A\widehat{x}_2(\lambda) + B\widehat{u}_2(\lambda) = 0, \\ \lambda \widehat{u}_2(\lambda) + C\widehat{u}_2(\lambda) - B^* \lambda \widehat{x}_2(\lambda) = -DD^* \widehat{u}(\lambda). \end{array} \right. \tag{3.16}$$

Since $\lambda^2 + A$ is invertible (Lax-Milgram lemma), we deduce from the first equation of the system (3.16) that

$$\widehat{x}_2 = -(\lambda^2 + A)^{-1} B \widehat{u}_2.$$

Substituting $\widehat{x}_2$ in the second equation of system (3.16), we get

$$(\lambda I + C + \lambda B^* (\lambda^2 + A)^{-1} B)\widehat{u}_2 = -DD^* \widehat{u}.$$

Noting that the invertibility of $\lambda I + C + \lambda B^*(\lambda^2 + A)^{-1}B$ follows from the invertibility of $\lambda I - \mathcal{A}_c$ we obtain

$$D^*\widehat{u}_2 = -[D^*(\lambda I + C + \lambda B^*(\lambda^2 + A)^{-1}B)^{-1}D]D^*\widehat{u}$$

and by section 3.3 or assumption (**H**) estimate (3.11) holds. Finally, the inequality, $\|z_0\|^2_{\mathcal{H}} \lesssim \|D^*u\|^2_{L^2(0,T;U)}$, implies that there is a constant $c_1(T)$ which depends only on T such that

$$E(0) - E(T) \geq c_1(T)E(0).$$

But it is well known (see for instance [9]) that the previous estimate is equivalent to (3.13). ■

3.5 Weaker decay

In the case of non exponential decay in the energy space we give sufficient conditions for weaker decay properties. The statement of our second result requires some notations.

Let $\mathcal{H}_1, \mathcal{H}_2$ be two Banach spaces such that

$$\mathcal{D}(\mathcal{A}_d) \subset \mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_2,$$

where

$$\|.\|_{\mathcal{D}(\mathcal{A}_d)} \sim \|.\|_{\mathcal{H}_1}$$

and

$$[\mathcal{H}_1; \mathcal{H}_2]_\theta = \mathcal{H} \tag{3.17}$$

for a fixed $\theta \in]0;1[$, where $[:;:]$ denotes the interpolation space (see for instance [51]).

Let $G : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be such that G is continuous, invertible, increasing on $\mathbb{R}_+$ and suppose that the function $x \longmapsto \dfrac{1}{x^{\frac{\theta}{1-\theta}}}G(x)$ is increasing on $(0;1)$.

Theorem 3.5.1 *Assume that the function G satisfies the above assumptions and that assumption* (**H**) *is satisfied or that* $D \in \mathcal{L}(U)$. *Then the following assertions hold true:*

1. *If for all non zero* $z_0 \in \mathcal{D}(\mathcal{A}_d)$, *the solution* $z_1(t) = (x_1(t), y_1(t), u_1(t))$ *of (3.8) satisfies the following observability estimate:*

$$\int_0^T \|D^*u_1(s)\|^2_U ds \geq c(T)\|z_0\|^2_{\mathcal{H}} G\left(\frac{\|z_0\|^2_{\mathcal{H}_2}}{\|z_0\|^2_{\mathcal{H}}}\right), \tag{3.18}$$

then we have

$$E(t) \lesssim \left[G^{-1}\left(\frac{1}{1+t}\right)\right]^{\frac{\theta}{1-\theta}} \|z_0\|^2_{\mathcal{D}(\mathcal{A}_d)}. \tag{3.19}$$

2. *If for all non zero* $z_0 \in \mathcal{D}(\mathcal{A}_d)$, *the solution* $z_1(t) = (x_1(t), y_1(t), u_1(t))$ *of (3.8) satisfies the following observability estimate:*

$$\int_0^T \|D^*u_1(s)\|^2_U ds \geq c(T)\|z_0\|^2_{\mathcal{H}_2}, \tag{3.20}$$

then we have

$$E(t) \lesssim \frac{1}{(1+t)^{\frac{\theta}{1-\theta}}} \|z_0\|^2_{\mathcal{D}(\mathcal{A}_d)}. \tag{3.21}$$

Proof.

1. Using the same arguments as in the proof of Theroem 3.4.1 we get from (3.18)

$$\forall z_0 \in \mathcal{D}(\mathcal{A}_d), \ \int_0^T \|D^* u(s)\|^2_U ds \geq c(T)\|z_0\|^2_{\mathcal{H}} G\left(\frac{\|z_0\|^2_{\mathcal{H}_2}}{\|z_0\|^2_{\mathcal{H}}}\right).$$

The sequel follows the proof of Theorem 2.4 of [9], therefore we give the outlines below. Using (3.17) and the interpolation inequality

$$\|z_0\|_{\mathcal{H}} \leq \|z_0\|^{1-\theta}_{\mathcal{H}_1} \|z_0\|^{\theta}_{\mathcal{H}_2}$$

we easily check

$$\frac{\|z_0\|^2_{\mathcal{H}_2}}{\|z_0\|^2_{\mathcal{H}}} \geq \frac{\|z_0\|_{\mathcal{H}}^{\frac{2-2\theta}{\theta}}}{\|z_0\|_{\mathcal{H}_1}^{\frac{2-2\theta}{\theta}}}, \ \forall z_0 \in \mathcal{D}(\mathcal{A}_d).$$

Consequently, using (3.9) and the fact that the function $t \mapsto \|z(t)\|_{\mathcal{H}}$ is nonincreasing and G is increasing we obtain the existence of a constant $K_1 > 0$ such that

$$\|z(T)\|^2_{\mathcal{H}} \leq \|z(0)\|^2_{\mathcal{H}} - K_1 \|z(0)\|^2_{\mathcal{H}} G\left(\frac{\|z(T)\|_{\mathcal{H}}^{\frac{2-2\theta}{\theta}}}{\|z(0)\|_{\mathcal{H}_1}^{\frac{2-2\theta}{\theta}}}\right).$$

Applying the same arguments on successive intervals $[kT, (k+1)T], \ k = 1, 2, ...$ we obtain the existence of a constant K_2 such that

$$\|z((k+1)T)\|^2_{\mathcal{H}} \leq \|z(kT)\|^2_{\mathcal{H}} - K_2 \|z(kT)\|^2_{\mathcal{H}} G\left(\frac{\|z((k+1)T)\|_{\mathcal{H}}^{\frac{2-2\theta}{\theta}}}{\|z(0)\|_{\mathcal{H}_1}^{\frac{2-2\theta}{\theta}}}\right), \ \forall z_0 \in \mathcal{D}(\mathcal{A}_d).$$

If we set $\mathcal{E}_k = G\left(\frac{\|z(kT)\|_{\mathcal{H}}^{\frac{2-2\theta}{\theta}}}{\|z(0)\|_{\mathcal{H}_1}^{\frac{2-2\theta}{\theta}}}\right)$, the previous inequality, the property of G and the fact that $t \mapsto \|z(T)\|_{\mathcal{H}}$ is nonincreasing then we get

$$\frac{\|z((k+1)T)\|^2_{\mathcal{H}}}{\|z(kT)\|^2_{\mathcal{H}}} \frac{\mathcal{E}_k}{\mathcal{E}_{k+1}} \mathcal{E}_k \leq \mathcal{E}_k - K_2 \mathcal{E}^2_{k+1}.$$

Equivalently, we have

$$\frac{\dfrac{1}{\left[\frac{\|z(kT)\|_{\mathcal{H}}^{\frac{2-2\theta}{\theta}}}{\|z(0)\|_{\mathcal{H}}^{\frac{2-2\theta}{\theta}}}\right]^{\frac{\theta}{1-\theta}}} G\left(\frac{\|z(kT)\|_{\mathcal{H}}^{\frac{2-2\theta}{\theta}}}{\|z(0)\|_{\mathcal{H}_1}^{\frac{2-2\theta}{\theta}}}\right)}{\dfrac{1}{\left[\frac{\|z((k+1)T)\|_{\mathcal{H}}^{\frac{2-2\theta}{\theta}}}{\|z(0)\|_{\mathcal{H}}^{\frac{2-2\theta}{\theta}}}\right]^{\frac{\theta}{1-\theta}}} G\left(\frac{\|z((k+1)T)\|_{\mathcal{H}}^{\frac{2-2\theta}{\theta}}}{\|z(0)\|_{\mathcal{H}_1}^{\frac{2-2\theta}{\theta}}}\right)} \mathcal{E}_{k+1} \leq \mathcal{E}_k - K_2 \mathcal{E}^2_{k+1}. \tag{3.22}$$

Combining (3.22) and the fact that the function $x \longmapsto \frac{1}{x^{\frac{\theta}{1-\theta}}} G(x)$ is increasing on $(0;1)$, we get

$$\mathcal{E}_{k+1} \leq \mathcal{E}_k - K_2 \mathcal{E}_{k+1}^2.$$

We thus deduce the existence of a constant $M > 0$ such that $\mathcal{E}_k \leq \frac{M}{k+1}$ and we finally get (3.19).

2. As for 1. the proof is similar to the second assertion of Theorem 2.4 of [9] which is based on Lemma 5.2 of [8] and is left to the reader.

■

3.6 Examples

Beam System

We consider the following beam equation:

$$\begin{cases} u_{tt}(x,t) + u^{(4)}(x,t) = 0, & 0 < x < 1, t \in [0,\infty) \\ \eta_t(t) + \beta\eta(t) - u_t(1,t) = 0, & 0 < x < 1, t \in [0,\infty) \\ u(0,t) = u'(0,t) = u''(1,t) = 0, & t \in [0,\infty) \\ u'''(1,t) = \eta(t) \end{cases} \tag{3.23}$$

with the initial conditions

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \eta(0) = \eta_0.$$

In this case

$$X = L^2(0,1), U = \mathbb{C}, V = \{u \in H^2(0,1) : u(0) = u'(0) = 0\},$$

$$\mathcal{D}(A) = \{u \in H^4(0,1) : u(0) = u'(0) = u''(1) = 0, u^{(3)}(1) = 0\},$$

$$a(u,v) = \int_0^1 \bar{u}^{(2)} v^{(2)} dx \, (u,v \in V), \; Au = u^{(4)} \, (u \in \mathcal{D}(A)),$$

$$B^* = \delta_1, B^*\varphi = \varphi(1) \, (\varphi \in V).$$

$$< B\eta, \varphi >_{V',V} = \bar{\eta}\varphi(1) \, (\eta \in \mathbb{C}, \varphi \in V),$$

and

$$\widehat{C} : \quad \begin{array}{c} \mathbb{C} \to \mathbb{C} \\ \eta \to -\beta\eta \end{array} ,$$

where β is a postive constant.

Note that $\widehat{C}$ is bounded, so we only need to find the observability inequality in order to deduce the type of stability of the system. Since $B \in \mathcal{L}(U, V')$ then $B\eta = \eta \cdot B1$, and since $B1 \in V'$ and $A \in \mathcal{L}(V, V')$ then there exists a unique $u_0 \in V$ such that $B1 = Au_0$. Indeed, it is easy to check that $u_0(x) = \frac{x^2}{2} - \frac{x^3}{6}$. Moreover, $C = 0$ and $D = \sqrt{\beta}$.

Remark that in this case

$$\mathcal{D}(\mathcal{A}_c) = \mathcal{D}(\mathcal{A}_d) = \{(u, v, \eta) \in V \times V \times \mathbb{C} : Au + B\eta \in L^2(0,1)\},$$

and

$$\mathcal{A}_c \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \begin{pmatrix} v \\ -Au - B\eta \\ B^*v \end{pmatrix}. \tag{3.24}$$

Note that since $\mathcal{D}(A_c)$ is compactly injected in $\mathcal{H}$, then $\mathcal{A}_c$ has a compact resolvent and thus its spectrum is discrete. In addition, since $\mathcal{A}_c$ is a skew-adjoint real operator, then its spectrum is constituted of pure imaginary conjugate eigenvalues. Now, let $\lambda = i\mu \in \sigma(\mathcal{A}_c)$ with U_μ an associated eigenvector then $\bar{\lambda} = -i\mu \in \sigma(\mathcal{A}_c)$ with $\bar{U}_\mu$ an associated eigenvector. Since the eigenvalues are conjugates , it is sufficient then to study $\mu \geq 0$.

Lemma 3.6.1 *The eigenvalues of $\mathcal{A}_c$ are algebraically simple. Moreover, $0 \in \sigma(\mathcal{A}_c)$ and for every $\lambda = i\mu \in \sigma(\mathcal{A}_c), \mu > 0$, μ satisfies the following characteristic equation,*

$$f(\mu) = \mu\sqrt{\mu} + \mu\sqrt{\mu}\cosh(\sqrt{\mu})\cos(\sqrt{\mu}) + \sin(\sqrt{\mu})\cosh(\sqrt{\mu}) - \cos(\sqrt{\mu})\sinh(\sqrt{\mu}) = 0. \tag{3.25}$$

Proof. First it is easy to see that 0 is a simple eigenvalue of $\mathcal{A}_c$ and that an associated eigenvector is $U = \eta(-u_0, 0, 1)^\top, \eta \in \mathbb{C}$.

Let $\lambda = i\mu \in \sigma(\mathcal{A}_c), \mu > 0$, and let $U = (u, v, \eta)^\top \in \mathcal{D}(\mathcal{A}_c)$ be a nonzero associated eigenvector. Then U satsifies

$$\mathcal{A}_c(u, v, \eta)^\top = \lambda(u, v, \eta)^\top,$$

which is equivalent to

$$\begin{cases} v = \lambda u \\ B^*v = \lambda\eta \\ -Au - \eta A u_0 = \lambda v = \lambda^2 u. \end{cases} \tag{3.26}$$

We then deduce that

$$A(u + \eta u_0) = -\lambda^2 u, B^*u = \lambda u(1) = \eta.$$

But as $U \in \mathcal{D}(\mathcal{A}_c)$, then $Au + B\eta = A(u + \eta u_0) \in L^2(0,1)$, which implies that $u + \eta u_0 \in \mathcal{D}(A)$ and that $u \in H^4(0,1)$ satisfies

$$u(0) = u'(0) = u''(1) = 0, u'''(1) = \eta. \tag{3.27}$$

However, $A(u + \eta u_0) = (u + \eta u_0)^{(4)} = u^{(4)}$, thus we need to solve

$$u^{(4)} = -\lambda^2 u = \mu^2 u,\ u(1) = \eta$$

with u satisfying (3.27). We deduce that u could be written as

$$u = c_1 \sin(\sqrt{\mu}x) + c_2 \sinh(\sqrt{\mu}x) + c_3 \cos(\sqrt{\mu}x) + c_4 \cosh(\sqrt{\mu}x),$$

with $C = (c_1, c_2, c_3, c_4)^\top$ satisfying

$$\widetilde{M}C = V_0 \tag{3.28}$$

where

$$\widetilde{M}=\begin{pmatrix}0&0&1&1\\1&1&0&0\\-\sin(\sqrt{\mu})&\sinh(\sqrt{\mu})&-\cos(\sqrt{\mu})&\cosh(\sqrt{\mu})\\-\cos(\sqrt{\mu})&\cosh(\sqrt{\mu})&\sin(\sqrt{\mu})&\sinh(\sqrt{\mu})\end{pmatrix},\ V_0=\begin{pmatrix}0\\0\\0\\\frac{\eta}{\mu\sqrt{\mu}}\end{pmatrix}.$$

We first remark that $\eta\neq 0$. Otherwise, since u satisfies $u^{(4)}=\mu^2 u$ and the boundary conditions $u(1)=u''(1)=u'''(1)=0$, then there exists a constant $c\in\mathbb{R}$ such that u is given by

$$u(x)=c(\sinh(\sqrt{\mu}(1-x))+\sin(\sqrt{\mu}(1-x)).$$

But $\cosh(\sqrt{\mu})+\cos(\sqrt{\mu})>0$, then $u'(0)=0$ implies that $c=0$ and hence $U=(u,\lambda u,\eta)^\top=0$ which is a contradiction.

Consequently, each eigenvalue of $\mathcal{A}_c$ is simple. In fact, suppose to the contrary that there exists $\mu\neq 0$ such that $\lambda=i\mu$ is not algebraically simple. Then as $\mathcal{A}_c$ is skew-adjoint, $\lambda=i\mu$ is not geometrically simple. Thus there exists at least two independent eigenvectors $U_i=(u_i,v_i,\eta_i), i=1,2$, corresponding to λ, and hence $U=\eta_2U_1-\eta_1U_2=(u,v,\eta)=(u,v,0)$ is an eigenvector which is impossible.

Going back to (3.28), we get from the first three equations,

$$c_2=-c_1,\ c_4=-c_3,\ c_3=-c_1\frac{\sin(\sqrt{\mu})+\sinh(\sqrt{\mu})}{\cos(\sqrt{\mu})+\cosh(\sqrt{\mu})}.$$

Therefore the last equation of (3.28) becomes

$$-\frac{2c_1(1+\cos(\sqrt{\mu})\cosh(\sqrt{\mu}))}{\cos(\sqrt{\mu})+\cosh(\sqrt{\mu})}=\frac{\eta}{\mu\sqrt{\mu}}.$$

As $\eta\neq 0$ then the determinant of $\widetilde{M}$ which is given by $\det(\widetilde{M})=-2\left(1+\cos(\sqrt{\mu})\cosh(\sqrt{\mu})\right)$ is nonzero and C is given by

$$C=\widetilde{M}^{-1}V_0=\frac{\eta}{2\mu\sqrt{\mu}(1+\cos(\sqrt{\mu})\cosh(\sqrt{\mu}))}\begin{pmatrix}-\cos(\sqrt{\mu})-\cosh(\sqrt{\mu})\\\cos(\sqrt{\mu})+\cosh(\sqrt{\mu})\\\sin(\sqrt{\mu})+\sinh(\sqrt{\mu})\\-\sin(\sqrt{\mu})-\sinh(\sqrt{\mu})\end{pmatrix}.$$

Substituting C in the condition $u(1)=\eta$, we finally deduce that μ satisfies the charateristic equation (3.25). ■

Now, we study the asymptotic behavior of the eigenvalues of $\mathcal{A}_c$.

Lemma 3.6.2 *There exists $k_0\in\mathbb{N}$ large enough such that for all $k\geq k_0$ there exists one and only one $\lambda_k=i\mu_k$ eigenvalue of $\mathcal{A}_c$ with $\sqrt{\mu_k}\in[k\pi,(k+1)\pi]$. Moreover, as $k\to\infty$, we have the following*

$$\sqrt{\mu_k}=\frac{\pi}{2}+k\pi+\frac{1}{k^3\pi^3}+o\left(\frac{1}{k^3}\right). \tag{3.29}$$

Let $U_{1,k}=(u_{1,k},\lambda_k u_{1,k},\eta_{1,k})$ be the associated normalized eigenvector. Then,

$$|\eta_{1,k}|^2=\frac{4}{k^4}+o\left(\frac{1}{k^4}\right). \tag{3.30}$$

Proof.

First step. Let $z = \sqrt{\mu}$ where $i\mu \in \sigma(\mathcal{A}_c)$ and $\mu > 0$. Then by (3.25), we have

$$f(z^2) = z^3 + \cosh z(z^3 \cos z + \sin z) - \cos z \sinh z = 0.$$

Replacing $\cosh z = \frac{e^z+e^{-z}}{2}$ and $\sinh z = \frac{e^z-e^{-z}}{2}$ in $f(z^2)$ and dividing by $\frac{z^3e^z}{2}$, we deduce that z satisfies $\tilde{f}(z) = 0$, where

$$\tilde{f}(z) = \cos z + \frac{\sin z - \cos z}{z^3} + 2e^{-z} + e^{-2z}\left(\cos z + \frac{\cos z}{z^3} + \frac{\sin z}{z^3}\right).$$

For z large enough we have

$$\tilde{f}(z) = \cos z + O(1/z^3).$$

It can be easily checked that for k large enough, $\tilde{f}$ doesn't admit any root outside the ball $B_k = B\left(z_k^0, \frac{1}{k^2}\right)$, with $z_k^0 = \frac{\pi}{2} + k\pi$. Then by Rouché's Theorem applied on B_k, we deduce that for k large enough there exists a unique root z_k of $\tilde{f}$ in $[k\pi, (k+1)\pi]$. Moreover, z_k satisfies

$$z_k = \frac{\pi}{2} + k\pi + \epsilon_k,$$

with $\epsilon_k = o(1)$. Since z_k satisifes $\tilde{f}(z_k) = 0$, then ϵ_k satisfies

$$\cos\left(\frac{\pi}{2} + k\pi + \epsilon_k\right) + \frac{\sin\left(\frac{\pi}{2} + k\pi + \epsilon_k\right) + o(1)}{k^3\pi^3 + o(k^3)} + O(e^{-z_k}) = 0.$$

Hence

$$-\sin(\epsilon_k) + \frac{\cos(\epsilon_k)}{k^3\pi^3} + o\left(\frac{1}{k^3}\right) = 0,$$

and thus

$$-k^3\epsilon_k + o(k^3\epsilon_k^2) + \frac{1}{\pi^3} + o(k^2\epsilon_k) + o(1) = 0,$$

which gives

$$\epsilon_k = \frac{1}{\pi^3 k^3} + o(1/k^3).$$

Therefore, (3.29) follows for $\mu_k = z_k^2$.

Second step. Set $\beta_k = \dfrac{\sin(z_k) + \sinh(z_k)}{\cos(z_k) + \cosh(z_k)}$. Then

$$\beta_k = \frac{e^{z_k} + 2\sin(z_k) - e^{-z_k}}{e^{z_k} + 2\cos(z_k) + e^{-z_k}} = 1 + o(e^{-z_k}). \tag{3.31}$$

By the proof of Lemma 3.6.1, the last component η_k^1 of U_k^1 is nonzero and thus

$$U_k = (u_k, iz_k^2 u_k, 1) = \frac{1}{\eta_{1,k}} U_{1,k}$$

is an associated eigenvector to iz_k^2 with u_k having the form,

$$u_k(x) = c_{1k}\sin(z_k x) + c_{2k}\sinh(z_k x) + c_{3k}\cos(z_k x) + c_{4k}\cosh(z_k x),$$

with

$$c_{2k} = -c_{1k},\ c_{4k} = -c_{3k},\ c_{3k} = -\beta_k c_{1k}.$$

It follows that

$$u_k(x) = c_{1k}\left[(\sin(z_k x) - \sinh(z_k x) - \cos(z_k x) + \cosh(z_k x)) + (\beta_k - 1)(-\cos(z_k x) + \cosh(z_k x))\right]. \tag{3.32}$$

In order to get the behavior of $\eta_k = \frac{1}{\|U_k\|}$, it is enough to compute the integral $\int_0^1 |u_k|^2 dx$. Indeed, multiplying $u_k^{(4)} = -\lambda^2 u_k = \mu^2 u_k$ by $\bar{u}_k$, integrating by parts and noting that

$$u_k(0) = u_k'(0) = 0,\ u_k(1) = u_k'''(1) = 1,$$

we obtain

$$\int_0^1 |u_k''|^2 dx = {\mu_k}^2 \int_0^1 u_k^2 dx - 1,$$

and hence

$$\|U_k\|^2 = \int_0^1 u_{kxx}^2 dx + {\mu_k}^2 \int_0^1 u_k^2 dx + 1 = 2{\mu_k}^2 \int_0^1 u_k^2 dx.$$

Since

$$\begin{aligned} 2z_k^3 c_{1k} &= \frac{-\cos z_k - \frac{e^{z_k}}{2}(1 + e^{-2z_k})}{1 + \frac{e^{z_k}}{2}\cos z_k (1 + e^{-2z_k})} \\ &= \frac{-1 + O(e^{-k})}{(-1)^{k+1}\sin \epsilon_k + O(e^{-k})}, \end{aligned}$$

we deduce that

$$c_{1k} = \frac{(-1)^k}{2} + o(1). \tag{3.33}$$

As

$$\int_0^1 (\sin(zx) - \sinh(zx) - \cos(zx) + \cosh(zx))^2 dx = \int_0^1 (\sin(zx) - \cos(zx))^2 dx + o(1) = 1 + o(1),$$

and

$$\int_0^1 (-\cos(zx) + \cosh(zx))^2 dx = \frac{e^{2z}}{8z} + o(\frac{e^{2z}}{8z}),$$

we consequently deduce due to (3.31), (3.32) and (3.33) that

$$\int_0^1 u_k^2(x) dx = \frac{1}{4} + o(1),\ \text{and}\ \|U_k\|^2 = \frac{k^4}{4} + o(k^4).$$

Hence (3.30) holds. ■

Proposition 3.6.3 *Let $U_1 = (u_1, v_1, \eta_1)^T$ be the solution of the conservative problem (3.24) with initial datum $U_0 \in \mathcal{D}(\mathcal{A}_c)$. Then there exists $T > 0$ and $c > 0$ depending on T such that*

$$\int_0^T |\eta_1(t)|^2 dt \geq c \|U_0\|^2_{D(A^{-1})}. \tag{3.34}$$

Proof. We arrange the elements of $\sigma(\mathcal{A}_c)$ in increasing order.

Let $J = \{i\mu : |\mu| < \mu_{k_0}\}$. Then $\sigma(\mathcal{A}_c) = J \cup \{i\mu_k : |k| \geq k_0\}$ and $(U_\mu)_{\mu\in J} \cup (U_{1,k})_{|k|\geq k_0}$ forms a Hilbert basis of $\mathcal{H}$. We may write

$$U_0 = \sum_{\mu\in J} u_0^\mu U_\mu + \sum_{|k|\geq k_0} u_0^{(k)} U_{1,k}.$$

Moreover,

$$\eta_1(t) = \sum_{\mu\in J} u_0^\mu e^{i\mu t}\eta_\mu + \sum_{|k|\geq k_0} u_0^{(k)} e^{i\mu_k t}\eta_{1,k}.$$

Note that $\mu_{k+1} - \mu_k \geq \frac{\pi}{2}$ for $|k| \geq k_0$. Set $\gamma_0 = \min\left\{\frac{\pi}{2}, \min\{|\mu - \mu'| : \mu \in J, \mu' \in J\}\right\}$. As $|\mu - \mu'| \geq \gamma_0 > 0$ for all consecutive $\mu \in \sigma(\mathcal{A}_c), \mu' \in \sigma(\mathcal{A}_c)$. Then using Ingham's inequality there exists $T > 2\pi\gamma_0 > 0$ and a constant $c > 0$ depending on T such that

$$\int_0^T |\eta_1(t)|^2 dt \geq c\left(\sum_{\mu\in J} |u_0^\mu \eta_\mu|^2 + \sum_{|k|\geq k_0} |u_0^{(k)}\eta_{1,k}|^2\right).$$

Due to Lemma 3.6.2, we have that $|\eta_{1,k}|^2 \sim \dfrac{1}{k^4}$. we deduce using Ingham's inequality the existence of $T > 0$ such that

$$\int_0^T |\eta_1|^2 dt \gtrsim \sum_{\mu\in J} |u_0^\mu|^2 |\mu|^{-2} + \sum_{|k|\geq k_0} \frac{|u_0^{(k)}|^2}{k^4}. \tag{3.35}$$

Therefore, we obtain (3.34) as required. ■

Theorem 3.6.4 *Let $U_0 \in \mathcal{D}(\mathcal{A}_d)$ and let U be the solution of the corresponding dissipative problem*

$$U_t = \mathcal{A}_d U,\ U(0) = U_0 \in \mathcal{D}(\mathcal{A}_d).$$

Then U satisfies,

$$\|U(t)\|^2 \lesssim \frac{1}{1+t}\|U_0\|^2_{\mathcal{D}(\mathcal{A}_d)}. \tag{3.36}$$

Proof. Since the operator $D \in \mathcal{L}(U)$, then Lemma 3.3.1 holds true.

Set $\mathcal{H}_1 = \mathcal{D}(\mathcal{A}_c)$ and $\mathcal{H}_2 = \mathcal{D}(\mathcal{A}_c^{-1})$, the dual of $\mathcal{D}(\mathcal{A}_c)$ obtained by means of the inner product in X. Then $\mathcal{H} = [\mathcal{H}_1; \mathcal{H}_2]_{1/2}$. By Proposition 3.6.3, we have

$$\int_0^T \|D^* u_1(s)\|_U^2 ds \geq c_T \|u_0\|^2_{\mathcal{H}_2}.$$

By Theorem 3.5.1 applied for $\theta = 1/2$, we therefore obtain (3.36). ■

Example on uniform stability

Consider the following system,

$$\begin{cases} u_{tt}(x,t) + u^{(4)}(x,t) + \alpha\theta_{xx}(x,t) = 0, & t \in [0,\infty), 0 < x < 1 \\ \theta_t(x,t) + \beta\theta(x,t) - \alpha u_{txx}(x,t) = 0, & t \in [0,\infty), 0 < x < 1 \\ u(0,t) = u(1,t) = u''(0,t) = u''(1,t) = 0 & t \in [0,\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \theta(x,0) = \theta_0(x), & 0 < x < 1 \end{cases} \tag{3.37}$$

with $\alpha > 0, \beta > 0$. Define the following spaces,

$$V = H^2(0,1) \cap H_0^1(0,1), X = U = L^2(0,1),$$

and the following operators,

$$\mathcal{D}(A) = \{u \in H^4(0,1) \cap H_0^1(0,1) : u_{xx}(0) = u_{xx}(1) = 0\},\ Au = u_{xxxx} \in L^2(0,1),$$

$$\begin{aligned} \widehat{C}:\ \ & L^2(0,1) \to L^2(0,1) \\ & \theta \to -\beta\theta. \end{aligned}$$

Remark that $\widehat{C}$ is a bounded operator on $L^2(0,1)$. Moreover, B and B^* are given by

$$\begin{aligned} B:\ \ & U \to V' \quad , \ B^* : V \to U \\ & \theta \to \alpha\theta_{xx} \qquad\quad u \to \alpha u_{xx}, \end{aligned}$$

and $D, D^* \in \mathcal{L}(U)$ with $D\theta = D^*\theta = \sqrt{\beta}\theta$. The norm defined on the energy space $\mathcal{H} = V \times X \times U$ is given by

$$\|(u,v,\theta)^\top\|_{\mathcal{H}}^2 = \int_0^1 |u_{xx}|^2 dx + \int_0^1 |v|^2 dx + \int_0^1 |\theta|^2 dx$$

We moreover have

$$\mathcal{D}(\mathcal{A}_d) = \mathcal{D}(\mathcal{A}_c) = \{(u,v,\theta)^\top \in V \times V \times U : u^{(4)} + \theta_{xx} \in L^2(\Omega)\}.$$

The associated conservative system is given by

$$\begin{cases} u_{tt}(x,t) + u^{(4)}(x,t) + \alpha\theta_{xx}(x,t) = 0, & t \in [0,\infty), 0 < x < 1 \\ \theta_t(x,t) - \alpha u_{txx}(x,t) = 0, & t \in [0,\infty), 0 < x < 1 \\ u(0,t) = u(1,t) = u''(0,t) = u''(1,t) = 0 & t \in [0,\infty) \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \theta(x,0) = \theta_0(x), & 0 < x < 1. \end{cases} \tag{3.38}$$

In the following proposition we prove that the solution u, θ of (3.38) satisfies the required observability inequality (assumption (O)), which is enough to deduce the exponential stability of (3.37) as $D \in \mathcal{L}(U)$.

Proposition 3.6.5 *Let $U_0 = (u_0, u_1, \theta_0)^\top \in \mathcal{H}$. Then the solution (u,θ) of (3.38) satisfies*

$$\int_0^T |\theta(t)|^2 dt \gtrsim \|U_0\|_{\mathcal{H}}^2. \tag{3.39}$$

Proof. Writing $(u_0, u_1, \theta_0)^\top \in \mathcal{D}(\mathcal{A}_c)$ with respect to the basis $(\sin(k\pi x))_{k\in\mathbb{N}^*}$ of $L^2(0,1)$, we have

$$u_0 = \sum_{k\in\mathbb{N}^*} u_k^0 \sin(k\pi x), u_1 = \sum_{k\in\mathbb{N}^*} u_k^1 \sin(k\pi x), \theta_0 = \sum_{k\in\mathbb{N}^*} \theta_k^0 \sin(k\pi x).$$

The solution (u,θ) of (3.38) is thus given by

$$u(t) = \sum_{k\in\mathbb{N}^*} u_k(t)\sin(k\pi x) \quad \text{and} \quad \theta(t) = \sum_{k\in\mathbb{N}^*} \theta_k(t)\sin(k\pi x).$$

By the second equation of (3.38),

$$\theta_k'(t) + \alpha k^2\pi^2 u_k'(t) = 0, \forall k \in \mathbb{N}^*.$$

Due to the initial conditions we deduce that

$$\theta_k(t) = -\alpha k^2\pi^2 u_k(t) + \theta_k^0 + \alpha k^2\pi^2 u_k^0.$$

Replacing u and θ in the first equation of (3.38), we deduce that

$$u_k''(t) + k^4\pi^4(1+\alpha^2)u_k(t) = \alpha k^2\pi^2(\theta_k^0 + \alpha k^2\pi^2 u_k^0), \forall k \in \mathbb{N}^*,$$

hence

$$u_k(t) = \frac{\alpha(\theta_k^0 + \alpha k^2\pi^2 u_k^0)}{k^2\pi^2(1+\alpha^2)} + c_1\cos(k^2\pi^2\sqrt{1+\alpha^2}t) + c_2\sin(k^2\pi^2\sqrt{1+\alpha^2}t),$$

where

$$c_1 = \frac{-\alpha\theta_k^0 + k^2\pi^2 u_k^0}{k^2\pi^2(1+\alpha^2)}, c_2 = \frac{u_k^1}{k^2\pi^2\sqrt{1+\alpha^2}},$$

obtained by the initial conditions $u_k(0) = u_k^0$, $u_k'(0) = u_k^1$ and $\theta_k(0) = \theta_k^0$.

Finally,

$$\theta_k(t) = \frac{1}{(1+\alpha^2)^{\frac{3}{2}}}[\sqrt{1+\alpha^2}(\theta_k^0 + \alpha k^2\pi^2 u_k^0) + \alpha\sqrt{1+\alpha^2}(\alpha\theta_k^0 - k^2\pi^2 u_k^0)\cos(\sqrt{1+\alpha^2}k^2\pi^2 t)$$
$$-\alpha(1+\alpha^2)u_k^1\sin(\sqrt{1+\alpha^2}k^2\pi^2 t)].$$

Set $T = \dfrac{2}{\sqrt{1+\alpha^2}\pi}$. Then,

$$\begin{aligned}|\theta_k(t)|^2 &= \frac{1}{(1+\alpha^2)^{\frac{5}{2}}\pi}\left[(2+\alpha^4)(\theta_k^0)^2 - 2\alpha(-2+\alpha^2)k^2\pi^2\theta_k^0 u_k^0 + \alpha^2(3k^4\pi^4(u_k^0)^2 + (1+\alpha^2)(u_k^1)^2)\right]\\ &= \frac{\alpha^2(1+\alpha^2)(u_k^1)^2}{(1+\alpha^2)^{\frac{5}{2}}\pi} + \begin{pmatrix} k^2u_k^0 & \theta_k^0\end{pmatrix} M \begin{pmatrix} k^2u_k^0 \\ \theta_k^0\end{pmatrix},\end{aligned}$$

where M is a square matrix given by

$$\begin{pmatrix} \frac{3\alpha^2\pi^3}{(1+\alpha^2)^{\frac{5}{2}}} & -\frac{\alpha(-2+\alpha^2)\pi}{(1+\alpha^2)^{\frac{5}{2}}} \\ -\frac{\alpha(-2+\alpha^2)\pi}{(1+\alpha^2)^{\frac{5}{2}}} & \frac{2+\alpha^4}{\pi(1+\alpha^2)^{\frac{5}{2}}} \end{pmatrix}.$$

But as

$$\det M = \frac{2\alpha^2\pi^2}{(1+\alpha^2)^3} > 0, \operatorname{trace} M = \frac{2+\alpha^4+3\alpha^2\pi^4}{\pi(1+\alpha^2)^{\frac{5}{2}}} > 0,$$

we deduce that $\lambda_{\min} \geq c > 0$ (where λ_{min} is the smallest eigenvalue of M) for some constant c independent of k and hence

$$\int_0^T |\theta_k(t)|^2 dt \geq T\left(\frac{\alpha^2(1+\alpha^2)(u_k^1)^2}{(1+\alpha^2)^{\frac{5}{2}}\pi} + \lambda_{\min}(M)(k^4(u_k^0)^2 + (\theta_k^0)^2)\right)$$

we get

$$\int_0^T |\theta(t)|^2 dt \gtrsim \sum_{k\in\mathbb{N}^*} (k^4(u_k^0)^2 + (\theta_k^0)^2 + (u_k^1)^2) \gtrsim \|U_0\|^2_{\mathcal{H}}.$$

We hence conclude (3.39) by denseness of $\mathcal{D}(\mathcal{A}_c)$ in $\mathcal{H}$. ■

Recall that the energy of a solution (u,θ) of (3.38) is defined by

$$E(t) = \frac{1}{2}\left(\int_0^1 |u_{xx}|^2 dx + \int_0^1 |u_t|^2 dx + \int_0^1 |\theta|^2 dx\right).$$

Theorem 3.6.6 *Let $U_0 \in \mathcal{H}$. Then there exists $\omega > 0$ such that the energy of the solution (u,θ) of (3.37) satisfies*

$$E(t) \lesssim e^{-\omega t} E(0), \forall t \in [0,+\infty). \tag{3.40}$$

Proof. By Proposition 3.6.5, assumption (O) holds true. Then (3.40) follows by applying Theorem 3.4.1. ■

Hybrid example-2D problem

Let Ω be a bounded domain of $\mathbb{R}^2$ whose boundary Γ satisfies

$$\Gamma = \Gamma_0 \cup \Gamma_1,\ \bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \phi, \text{ and } \operatorname{meas} \Gamma_0 \neq 0.$$

We assume moreover that there exists a point $x_0 \in \mathbb{R}^2$ such that

$$\Gamma_0 = \{x \in \Gamma : m(x).\nu \leq 0\},\ \Gamma_1 = \{x \in \Gamma : m(x).\nu \geq \omega > 0\},$$

for some constant $\omega > 0$, where $m(x) = x - x_0$ and $\nu = \nu(x)$ denotes the unit outward normal vector at $x \in \Gamma$. Denote by $R = \|m\|_\infty = \sup_{x\in\Omega}\|m(x)\|$.

Consider the following system,

$$(P_b) \qquad \begin{cases} y_{tt}(x,t) - \Delta y(x,t) = 0, & x \in \Omega, t > 0,\\ y(x,t) = 0, & x \in \Gamma_0, t > 0,\\ a y_{tt}(x,t) + \partial_\nu y(x,t) + \eta(x,t) = 0, & x \in \Gamma_1, t > 0,\\ \eta_t(x,t) - y_t(x,t) + b\eta(x,t) = 0, & x \in \Gamma_1, t > 0,\\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x), & x \in \Omega,\\ \eta(x,0) = \eta_0(x) & x \in \Gamma_1, \end{cases}$$

where a and b are two positive constants. In order to justify that the system could be written in the proposed general form, we introduce a proper functional setting. Let

$$X = L^2(\Omega) \times L^2(\Gamma_1)$$

endowed with the inner product,

$$\left\langle (y,\xi)^\top, (\tilde{y},\tilde{\xi})^\top \right\rangle_X = \int_\Omega < y, \tilde{y} > dx + \frac{1}{a}\int_{\Gamma_1} < \xi, \tilde{\xi} > ds$$

and

$$W = \{y \in H^1(\Omega) : y = 0 \text{ on } \Gamma_0\} = H^1_{\Gamma_0}(\Omega),\ U = L^2(\Gamma_1).$$

Define also V by

$$V = \{(y,\xi) \in W \times L^2(\Gamma_1) : ay = \xi \text{ on } \Gamma_1\},$$

and the operator $(A, \mathcal{D}(A))$ by

$$A(y,\xi)^\top = (-\Delta y, \partial_\nu y|_{\Gamma_1})^\top$$

with

$$\mathcal{D}(A) = \{x = (y,\xi)^\top \in V : y \in H^2(\Omega)\}.$$

We can easily check using Lax-Milgram lemma that $(A \pm iI)$ are surjective. In addition, since A is symmetric we deduce that A is self-adjoint. The corresponding form $\tilde{a}$ is given by

$$\tilde{a}(u,\tilde{u}) = \int_\Omega < y_x, \tilde{y}_x > dx,\ u = (y,\xi)^\top \in V, \tilde{u} = (\tilde{y},\tilde{\xi})^\top \in V.$$

In addition, we define for every $\eta \in U$ and $(y,\xi)^\top \in V$ the operators B and B^* by

$$B\eta = (0,\eta)^\top,\ B^*(y,\xi)^\top = y|_{\Gamma_1}.$$

The operator $C = 0$ and the operator $\widehat{C}$ is given by

$$\widehat{C}\eta = -b\eta,\ \eta \in L^2(\Gamma_1).$$

Hence the system (P_b) can be written in the form of system (3.2).

Accordingly, we define the energy space

$$\mathcal{H} = V \times L^2(\Omega) \times L^2(\Gamma_1)^2,$$

endowed with the inner product

$$(u,\tilde{u})_{\mathcal{H}} = \int_\Omega < y_x, \tilde{y_x} > dx + \int_\Omega < z, \tilde{z} > dx + \frac{1}{a}\int_{\Gamma_1} < \xi, \tilde{\xi} > ds + \int_{\Gamma_1} < \eta, \tilde{\eta} > ds,$$

where $u = (y,\zeta,z,\xi,\eta)$, $\tilde{u} = (\tilde{y},\tilde{\zeta},\tilde{z},\tilde{\xi},\tilde{\eta}) \in \mathcal{H}$, and $< .,. >$ represents the Hermitian product in $\mathbb{C}$. The associated norm will be denoted by $\|\cdot\|_{\mathcal{H}}$. Moreover, $(\mathcal{A}_d, \mathcal{D}(\mathcal{A}_d))$ is then given by

$$\mathcal{A}_d u = (z, \xi, \Delta y, -\partial_\nu y - \eta, z|_{\Gamma_1} - b\eta),\ \forall u = (y,\zeta,z,\xi,\eta) \in \mathcal{D}(\mathcal{A}_d),$$

with

$$\mathcal{D}(\mathcal{A}_d) = \{u = (y, \zeta, z, \xi, \eta) \in \mathcal{H} : y \in H^2(0,1), z \in W, \zeta = ay|_{\Gamma_1} \xi = az|_{\Gamma_1}\}.$$

Hence, the previous problem (P_b) is formally equivalent to

$$u_t = \mathcal{A}_d u, \;\; u(0) = u_0, \tag{3.41}$$

where $u_0 = (y_0, ay_0|_{\Gamma_1}, y_1, ay_1|_{\Gamma_1}, \eta_0)$. The energy of the system (P_b) is given by

$$E(t) = \frac{1}{2}\left(\int_\Omega |y_t|^2 dx + \int_\Omega |\nabla y|^2 dx + \frac{1}{a}\int_{\Gamma_1} |y_t|^2 ds + \int_{\Gamma_1} |\eta^2| ds\right),$$

and its derivative

$$\frac{d}{dt}E(t) = -b\int_{\Gamma_1} |\eta^2| ds.$$

The corresponding conservative system is defined by

$$(P_0) \qquad \begin{cases} y_{tt}(x,t) - \Delta y(x,t) = 0, & x \in \Omega, t > 0, \\ y(x,t) = 0, & x \in \Gamma_0, t > 0, \\ ay_{tt}(x,t) + \partial_\nu y(x,t) + \eta(x,t) = 0, & x \in \Gamma_1, t > 0, \\ \eta_t(x,t) - y_t(x,t) = 0, & x \in \Gamma_1, t > 0, \\ y(x,0) = y_0(x),\, y_t(x,0) = y_1(x), & x \in \Omega, \\ \eta(x,0) = \eta_0(x) & x \in \Gamma_1. \end{cases}$$

The initial value problem associated to the conservative system (P_0) is given by

$$u_t = \mathcal{A}_c u, \;\; u(0) = u_0, \tag{3.42}$$

where

$$\mathcal{A}_c u = (z, \xi, \Delta y, -\partial_\nu y - \eta, z|_{\Gamma_1}),\ \forall u = (y, \zeta, z, \xi, \eta) \in \mathcal{D}(\mathcal{A}_c),\;\; \mathcal{D}(\mathcal{A}_c) = \mathcal{D}(\mathcal{A}_d).$$

As the operators D and D^* given by,

$$D\eta = D^*\eta = \sqrt{b}\eta,\ \eta \in L^2(\Gamma_1),$$

are bounded, Lemma 3.3.1 holds true. Thus in order to deduce the polynomial stability of the solution of (3.41), it is sufficient to check that the solution u_1 of (3.42) satisfies the observability inequality (O),

$$b\int_0^T \int_{\Gamma_1} |\eta_1^2| \gtrsim \|u_0\|^2_{\mathcal{D}(\mathcal{A}_c^{-2})},$$

where $\mathcal{D}(\mathcal{A}_c^{-2})$ denotes throughout the example the space $(\mathcal{D}(\mathcal{A}_c^2))'$.
We first state the following proposition.

Lemma 3.6.7 *Let $u_0 = (y_0, \zeta_0, z_0, \xi_0, \eta_0)^\top \in \mathcal{H}$ and let $u_1 = (y_1, \zeta_1, z_1, \xi_1, \eta_1)^\top$ be the corresponding solution of the problem* (3.42)*. Then there exists $c_T > 0$ such that*

$$\int_0^T \int_{\Gamma_1} |\eta_1^2| \geq c_T \|u_0\|^2_{\mathcal{D}(\mathcal{A}_c^{-2})}. \tag{3.43}$$

Proof. First step. Let $v_0 \in \mathcal{D}(\mathcal{A}_c)$ and $v = (y, \zeta, z, \xi, \eta)^\top$ be a solution of

$$v_t = \mathcal{A}_c v, \ v(0) = v_0. \tag{3.44}$$

Then there exist two positive constants $C_i > 0, i = 1, 2$ such that

$$(T - C_1)\|v_0\|_{\mathcal{H}}^2 \leq C_2 \left(\int_0^T \int_{\Gamma_1} |y_t|^2 + \int_0^T \int_{\Gamma_1} |\partial_\nu y|^2 + \int_0^T \int_{\Gamma_1} |\eta|^2 \right), \tag{3.45}$$

for all $T > 0$.

Indeed, for $v_0 \in \mathcal{D}(\mathcal{A}_c)$, we have

$$\begin{aligned} \int_0^T \int_\Omega y_{tt}(2m \cdot \nabla y) &= -\int_0^T \int_\Omega y_t(2m \cdot \nabla y_t) + \left[\int_\Omega y_t 2m \cdot \nabla y \right]_0^T \\ &= 2\int_0^T \int_\Omega |y_t|^2 - \int_0^T \int_\Gamma (m \cdot \nu)|y_t|^2 + \left[\int_\Omega y_t 2m \cdot \nabla y \right]_0^T, \end{aligned} \tag{3.46}$$

and

$$\begin{aligned} \int_0^T \int_\Omega \Delta y(2m \cdot \nabla y) &= -\int_0^T \int_\Omega \nabla y \cdot \nabla(2m \cdot \nabla y) + \int_0^T \int_\Gamma \partial_\nu y(2m \cdot \nabla y) \\ &= -\int_0^T \int_\Gamma (m \cdot \nu)|\nabla y|^2 + \int_0^T \int_\Gamma \partial_\nu y(2m \cdot \nabla y). \end{aligned} \tag{3.47}$$

Finally, multiplying the wave equation by $2m \cdot \nabla y$ and substracting (3.47) from (3.46) leads to,

$$\begin{aligned} 2\int_0^T \int_\Omega |y_t|^2 - \int_0^T \int_{\Gamma_1} (m \cdot \nu)|y_t|^2 + \int_0^T \int_\Gamma (m \cdot \nu)|\nabla y|^2 \\ + \left[\int_\Omega y_t 2m \cdot \nabla y \right]_0^T - \int_0^T \int_\Gamma \partial_\nu y(2m \cdot \nabla y) = 0. \end{aligned} \tag{3.48}$$

Multiplying the wave equation equation by y we obtain

$$-\int_0^T \int_\Omega |y_t|^2 + \int_0^T \int_\Omega |\nabla y|^2 + \left[\int_\Omega y_t y \right]_0^T - \int_0^T \int_\Gamma (\nu \cdot \nabla y) y = 0. \tag{3.49}$$

As

$$\int_0^T \int_\Gamma \partial_\nu y(2m \cdot \nabla y) = 2\int_0^T \int_\Gamma (m.\nu)(\partial_\nu y)^2 + 2\int_0^T \int_\Gamma (m.\tau)(\partial_\nu y \partial_\tau y),$$

then taking into consideration the Dirichlet condition on Γ_0, we get

$$\begin{aligned} \int_0^T \int_\Gamma \partial_\nu y(2m \cdot \nabla y) - \int_0^T \int_\Gamma (m \cdot \nu)|\nabla y|^2 &= \int_0^T \int_\Gamma (m.\nu)(\partial_\nu y)^2 - \int_0^T \int_{\Gamma_1} (m.\nu)(\partial_\tau y)^2 \\ &\quad +2\int_0^T \int_{\Gamma_1} (m.\tau)(\partial_\nu y \partial_\tau y). \end{aligned}$$

Due to the geometric conditions imposed on Γ, we have

$$\begin{aligned}\int_0^T\int_\Gamma \partial_\nu y(2m\cdot\nabla y)-\int_0^T\int_\Gamma (m\cdot\nu)|\nabla y|^2 &\leq \int_0^T\int_{\Gamma_1}(m.\nu)(\partial_\nu y)^2+\frac{R^2}{\omega}\int_0^T\int_{\Gamma_1}(\partial_\nu y)^2 \\ &\leq (R+\frac{R^2}{\omega})\int_0^T\int_{\Gamma_1}(\partial_\nu y)^2.\end{aligned} \tag{3.50}$$

Hence (3.48) leads to

$$2\int_0^T\int_\Omega |y_t|^2+\left[\int_\Omega y_t 2m\cdot\nabla y\right]_0^T \leq \int_0^T\int_{\Gamma_1}(m\cdot\nu)|y_t|^2+(R+\frac{R^2}{\omega})\int_0^T\int_{\Gamma_1}(\partial_\nu y)^2. \tag{3.51}$$

Adding (3.49) to (3.51), we obtain

$$\begin{aligned}\int_0^T\int_\Omega |y_t|^2+\int_0^T\int_\Omega|\nabla y|^2+\left[\int_\Omega y_t 2m\cdot\nabla y\right]_0^T+\left[\int_\Omega y_t y\right]_0^T-\int_0^T\int_\Gamma(\nu\cdot\nabla y)y \\ \leq \int_0^T\int_{\Gamma_1}(m\cdot\nu)|y_t|^2+(R+\frac{R^2}{\omega})\int_0^T\int_{\Gamma_1}(\partial_\nu y)^2.\end{aligned} \tag{3.52}$$

Note moreover that $\left[\int_\Omega y_t 2m\cdot\nabla y\right]_0^T+\left[\int_\Omega y_t y\right]_0^T \gtrsim -E(0)$, and

$$\int_0^T\int_{\Gamma_1}\partial_\nu y y\leq \frac{1}{2\epsilon}\int_0^T\int_{\Gamma_1}(\partial_\nu y)^2+\frac{\epsilon}{2}\int_0^T\int_{\Gamma_1}y^2\leq\frac{1}{2\epsilon}\int_0^T\int_{\Gamma_1}(\partial_\nu y)^2+\frac{c_p\epsilon}{2}\int_0^T\int_\Omega|\nabla y|^2.$$

We deduce that for $\epsilon>0$ chosen small enough there exists $C>0$ such that

$$\begin{aligned}(T-C_1)\|v_0\|^2_{\mathcal{H}}-\frac{1}{a}\int_0^T\int_{\Gamma_1}|y_t|^2-\int_0^T\int_{\Gamma_1}|\eta|^2 &\leq (R+\frac{R^2}{\omega}+\frac{1}{2\epsilon})\int_0^T\int_{\Gamma_1}(\partial_\nu y)^2 \\ &\quad+\int_0^T\int_{\Gamma_1}(m\cdot\nu)|y_t|^2,\end{aligned} \tag{3.53}$$

which leads to the required result (3.45).

Second step. Let $\alpha>0$ and set

$$\eta_1=\frac{1}{a}(-\partial_\nu y-\eta)+2\alpha z+\alpha^2\eta$$

We have the following expression for $|\eta_1|^2$ on Γ_1,

$$|\eta_1|^2=\frac{1}{a^2}|\partial_\nu y|^2+4\alpha^2|z|^2+\frac{(a\alpha^2-1)^2}{a^2}|\eta|^2-\frac{4\alpha}{a}\partial_\nu yz+\frac{2}{a^2}(1-a\alpha^2)\partial_\nu y\eta+\frac{4\alpha}{a}(a\alpha^2-1)z\eta.$$

By the boundary condition on Γ_1, $\eta_t=z$ and $\partial_\nu y=-\eta-a\eta_{tt}$, we get

$$|\eta_1|^2=\frac{1}{a^2}|\partial_\nu y|^2+4\alpha^2|z|^2+\frac{(-1+a\alpha^2)(1+a\alpha^2)}{a^2}|\eta|^2+4\alpha^3\eta\eta_t+4\alpha\eta_t\eta_{tt}+\frac{2\left(-1+a\alpha^2\right)}{a}\eta\eta_{tt}.$$

Thus

$$\begin{aligned}\int_0^T \int_{\Gamma_1} |\eta_1|^2 ds \quad = \quad & \int_0^T \int_{\Gamma_1} [\frac{1}{a^2}|\partial_\nu y|^2 + (4\alpha^2 - \frac{2(-1+a\alpha^2)}{a})|z|^2 + \frac{(-1+a\alpha^2)(1+a\alpha^2)}{a^2}|\eta|^2]ds \\ & +2\alpha^3 \int_{\Gamma_1} (\eta(T)^2 - \eta(0)^2)ds + 2\alpha \int_{\Gamma_1} (\eta_t(T)^2 - \eta_t(0)^2)ds + \\ & \frac{2(-1+a\alpha^2)}{a} \int_{\Gamma_1} \eta(T)\eta_t(T)ds - \frac{2(-1+a\alpha^2)}{a} \int_{\Gamma_1} \eta(0)\eta_t(0))ds.\end{aligned} \tag{3.54}$$

Choosing α large enough, we get

$$w_1 = \frac{1}{a^2} > 0,\ w_2 = 4\alpha^2 - \frac{2(-1+a\alpha^2)}{a} = \frac{2(1+a\alpha^2)}{a} > 0, w_3 = \frac{(-1+a\alpha^2)(1+a\alpha^2)}{a^2} > 0.$$

In addition, (3.54) implies that

$$\int_0^T \int_{\Gamma_1} |\eta_1|^2 ds \geq \int_0^T \int_{\Gamma_1} [w_1|\partial_\nu y|^2 + w_2|z|^2 + w_3|\eta|^2]ds - K_{a,\alpha}\|v_0\|^2_{\mathcal{H}},$$

for some constant $K_{a,\alpha} \geq 0$ independent of T.

Combining the previous inequality with (3.45), we deduce the existence of $c_1 > 0$ such that

$$\int_0^T \int_{\Gamma_1} |\eta_1|^2 ds \geq c_1(T - C)\|v_0\|^2_{\mathcal{H}} - K_{a,\alpha}\|v_0\|^2_{\mathcal{H}}.$$

Finally, choosing T large enough, we obtain

$$\int_0^T \int_{\Gamma_1} |\eta_1|^2 ds = \int_0^T \int_{\Gamma_1} \left|\frac{1}{a}(-\partial_\nu y - \eta) + 2\alpha z + \alpha^2 \eta\right|^2 ds \geq c_2\|v_0\|^2_{\mathcal{H}}, \tag{3.55}$$

for some positve constant c_2 depending on T.

Last step. Let $u_0 \in D(\mathcal{A}_d)$ and let $u_1 = (y_1, \zeta_1, z_1, \xi_1, \eta_1)^\top$ be the corresponding solution of (3.42), then

$$v = (y, \zeta, z, \xi, \eta)^\top = [(\mathcal{A}_c + \alpha I)^2]^{-1} u,$$

is a solution of (3.44) where $v_0 = [(\mathcal{A}_c + \alpha I)^2]^{-1} u_0 \in \mathcal{D}(\mathcal{A}_c)$. Since $(\mathcal{A}_c + \alpha I)^2 = \mathcal{A}_c^2 + 2\alpha\mathcal{A}_c + \alpha^2 I$, the last component η_1 of u_1 is given by

$$\eta_1 = \frac{1}{a}(-\partial_\nu y - \eta) + 2\alpha z + \alpha^2 \eta,$$

thus by the two previous steps we get (3.55). Noting that $\|u_0\|_{\mathcal{D}(\mathcal{A}_c)} \sim \|v_0\|_{\mathcal{H}}$, we consequently deduce that (3.43) holds for all $u_0 \in \mathcal{D}(\mathcal{A}_c)$. ■

Theorem 3.6.8 *Let $u_0 \in \mathcal{D}(\mathcal{A}_d)$ and let u be the solution of (3.41). Then u satisfies,*

$$\|u(t)\|^2 \lesssim \frac{1}{(1+t)^{\frac{1}{2}}} \|u_0\|^2_{\mathcal{D}(\mathcal{A}_d)}. \tag{3.56}$$

Proof. Since the operator $D \in \mathcal{L}(U)$, Lemma 3.3.1 holds true.

Set $\mathcal{H}_1 = D(\mathcal{A}_c)$ and $\mathcal{H}_2 = D(\mathcal{A}_c^{-2})$. Then $\mathcal{H} = [\mathcal{H}_1; \mathcal{H}_2]_{1/3}$. By Lemma 3.6.7, we have

$$\int_0^T \|D^* u_1(s)\|_U^2 ds \geq c_T \|u_0\|_{\mathcal{H}_2}^2.$$

By Theorem 3.5.1 applied for $\theta = 1/3$, we therefore obtain (3.56). ■

Remark 3.6.9 *Using the same method we get an analogous result for the one dimensional problem. we can also get the observability inequality by a spectrum analysis and that was already done in the paper [34], where the authors obtained an optimal decay, thus we expect the decay in the two dimensional case to be optimal as well.*

Remark 3.6.10 *Consider the following system studied in chapter 1 (see [2])*

$$\begin{cases} y_{tt}(x,t) - y_{xx}(x,t) & = & 0, & 0 < x < 1, t > 0, \\ y(0,t) & = & 0, & t > 0, \\ y_x(1,t) + (\eta(t), C_0)_{\mathbb{C}^n} & = & 0, & t > 0, \\ \eta_t(t) - B_0\eta(t) - C_0 y_t(1,t) & = & 0, & t > 0, \end{cases} \tag{3.57}$$

and

$$y(x,0) = y_0(x), y_t(x,0) = y_1(x), \eta(0) = \eta_0, 0 < x < 1,$$

where $B_0 \in M_n(\mathbb{C})$, $C_0 \in \mathbb{C}^n$ are given. System (3.57) can be written in the form (3.1) where $V = \{y \in H^1(0,1) : y(0) = 0\}$, $X = L^2(0,1)$ and $U = \mathbb{C}^n$. In this case, $\widehat{C} = B_0$ is a bounded operator and $B\eta = (\eta, C_0)\delta_1$ for all $\eta \in \mathbb{C}^n$. Indeed, since $\widehat{C}$ is bounded then it is enough to verify assumption (O). Assumption (O) was verified in [2] and the polynomial stability of (3.57) was deduced. In particular, for $n = 1$ we obtain the system studied in [53], where a polynomial decay is proved using a mutltiplier method. The polynomial decay can be also obtained by proving an observability inequality for the solutions of the corresponding conservative system which is exactly what has been verified in [2], thus applying the appraoch intoduced in this chapter .

3.6.1 Unbounded example

Consider the following system

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + w(x,t) = 0, & t \in [0,\infty),\ 0 < x < 1 \\ w_t(x,t) - iw_{xx}(x,t) + w(\xi,t)\delta_\xi - u_t(x,t) = 0, & t \in [0,\infty),\ 0 < x < 1 \\ u(0,t) = u(1,t) = w(0,t) = w(1,t) = 0, & t \in [0,\infty), \\ u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x),\ w(x,0) = w_0, & 0 < x < 1, \end{cases} \tag{3.58}$$

where $\xi \in (0,1)$. Define the following spaces and operators:

$$X = U = L^2(0,1), V = H_0^1(0,1), U = L^2(0,1), W = \mathbb{C},$$

$$A : u \in D(A) \to -u_{xx} \in L^2(0,1),\ D(A) = H^2(0,1) \cap H_0^1(0,1),$$

and $B = B^* = I_U = I_{L^2(0,1)}$. In addition,

$$D : \eta \in \mathbb{C} \to \eta\delta_\xi \in (D(C))',\ D^* : w \in D(C) \to w(\xi) \in \mathbb{C},$$

and

$$C : w \in D(C) \to -iw_{xx} \in L^2(0,1),\ D(C) = H^2(0,1) \cap H_0^1(0,1).$$

The operator $\widehat{C}$ is thus given by

$$\widehat{C}w = iw_{xx} - w(\xi)\delta_\xi$$

with

$$D(\widehat{C}) = \{w \in H_0^1(0,1) \cap [H^2(0,\xi) \cap H^2(\xi,1)] : i[w_x]_\xi = i[w_x(\xi^+) - w_x(\xi^-)] = w(\xi)\}.$$

As the operator D is unbounded, we need to verify that the problem satisfies assumption (H) as well as the asumption (O) for conservative problem. In this case we have

$$\mathcal{D}(\mathcal{A}_d) = D(A) \times V \times D(\widehat{C})$$

and

$$\mathcal{D}(\mathcal{A}_c) = D(A) \times V \times D(C).$$

In order to verify the assumption (H), we proceed by finding the transfer function, for this purpose we recall that $u_2 = u - u_1$ and $w_2 = w - w_1$ satisfies (3.15) which is in this case

$$\begin{cases} \partial_{tt}u_2 - \partial_{xx}u_2(x,t) + w_2(x,t) = 0, & t \in [0,\infty),\ 0 < x < 1 \\ \partial_t w_2(x,t) - i\partial_{xx}w_2(x,t) - \partial_t u_2 = w(\xi,t)\delta_\xi, & t \in [0,\infty),\ 0 < x < 1 \\ u_2(0,t) = u_2(1,t) = w_2(0,t) = w_2(1,t) = 0, & t \in [0,\infty), \\ u_2(x,0) = 0,\ \partial_t u_2(x,0) = 0,\ w_2(x,0) = 0, & 0 < x < 1, \end{cases} \tag{3.59}$$

and

$$\begin{cases} \partial_{tt}u_1 - \partial_{xx}u_1(x,t) + w_1(x,t) = 0, & t \in [0,\infty),\ 0 < x < 1 \\ \partial_t w_1(x,t) - i\partial_{xx}w_1(x,t) - \partial_t u_1 = 0, & t \in [0,\infty),\ 0 < x < 1 \\ u_1(0,t) = u_1(1,t) = w_1(0,t) = w_1(1,t) = 0, & t \in [0,\infty), \\ u_1(x,0) = u_0,\ \partial_t u_1(x,0) = u_1\ w_2(x,0) = w_0, & 0 < x < 1, \end{cases} \tag{3.60}$$

Verifying the assumption (H) is equivelant to verifying (see [9, Proposition 3.2] for more details)

$$|w_2(\xi,t)|^2 \lesssim |w(\xi,t)|^2.$$

For this purpose, we state the following proposition.

Proposition 3.6.11 *Let $(u_2, w_2) = (u - u_1, w - w_1)$ be the solution of (3.59). Then w_2 verifies*

$$|w_2(\xi,t)|^2 \le |w(\xi,t)|^2.$$

Proof. Let $\lambda = 1 + i\eta$ and consider $\hat{u}_2, \hat{w}_2$ the Laplace transforms of u_2 and w_2 respectively. Then $\hat{u}_2$ and $\hat{w}_2$ satisfies (3.16) given by

$$\begin{cases} \lambda^2\hat{u}_2(x,\lambda) - \partial_{xx}\hat{u}_2(x,\lambda) + \hat{w}_2(x,\lambda) = 0, \\ \lambda\hat{w}_2(x,\lambda) - i\partial_{xx}\hat{w}_2(x,\lambda) - \lambda\hat{u}_2 = \hat{w}(\xi,\lambda)\delta_\xi, \end{cases} \tag{3.61}$$

The problem reduces to studying $\hat{u}_2$ and $\hat{w}_2$ solutions of

$$\begin{cases} \lambda^2\widehat{u}_2 - \partial_{xx}\widehat{u}_2 + \widehat{w}_2 = 0 \\ \lambda\widehat{w}_2 - i\partial_{xx}\widehat{w}_2 - \lambda\widehat{u}_2 = -i\delta_\xi \end{cases} \tag{3.62}$$

with

$$\hat{u_2}(0) = \hat{u_2}(1) = 0, \hat{w_2}(0) = \hat{w_2}(1) = 0, [\partial_x \hat{w_2}]_\xi = 1, [\hat{w_2}]_\xi = 0,$$

and proving the existence of $C_\beta > 0$

$$|\hat{w_2}(\xi, \lambda|) \leq C_\beta, \forall \lambda = \beta + iy, y \in \mathbb{R}.$$

First, we set

$$\widehat{w}_2 = \widehat{w}_3 + \widehat{w}_4,$$

where

$$\lambda \widehat{w}_3 - i\partial_{xx}\widehat{w}_3 = -i\delta_\xi, \tag{3.63}$$

with

$$\widehat{w}_3(0) = \widehat{w}_3(1) = 0, [\partial_x \widehat{w}_3]_\xi = 1, [\widehat{w}_3]_\xi = 0. \tag{3.64}$$

and

$$\lambda \widehat{w}_4 - i\partial_{xx}\widehat{w}_4 = \lambda \widehat{u}_2. \tag{3.65}$$

with

$$\widehat{w}_4(0) = \widehat{w}_4(1) = 0. \tag{3.66}$$

Let $\beta > 0$ be fixed. It is required then to prove that

$$|\widehat{w}_3(\xi, \lambda)| \leq C_{1\beta}, \ |\widehat{w}_4(\xi, \lambda)| \leq C_{2\beta}, \forall \lambda = \beta + iy, y \in \mathbb{R}.$$

We start by writing the expression of $\widehat{w}_3$,

$$\widehat{w}_3(x, \lambda) = -\sum_{k=1}^{+\infty} \frac{\sqrt{2}\sin(k\pi\xi)}{k^2\pi^2 - i\lambda}\sqrt{2}\sin(k\pi x) = -2\sum_{k=1}^{+\infty} \frac{|\sin(k\pi\xi)|^2}{k^2\pi^2 - i\lambda}.$$

For simplicity we consider $\lambda = 1 \pm i\pi^2 y^2$.

$$|\widehat{w}_3(\xi, \lambda)| \lesssim \sum_{k=1}^{+\infty} \frac{1}{|(k^2 \pm y^2)\pi^2 - i|}.$$

We first give an estimate for $\lambda = 1 + i\pi^2 y^2$,

$$|\widehat{w}_3(\xi, \lambda)| \lesssim \sum_{k=1}^{+\infty} \frac{1}{(k^2 + y^2)\pi^2} \leq \frac{1}{6}.$$

For $\lambda = 1 - i\pi^2 y^2$, we have

$$|\widehat{w}_3(\xi, \lambda)| \leq \frac{2}{\pi^2}\left(\sum_{1\leq k\leq E(y)-1} \frac{1}{y^2 - k^2} + 2\pi^2 + \sum_{E(y)+2\leq k} \frac{1}{k^2 - y^2}\right).$$

But

$$\sum_{1\leq k\leq E(y)-1} \frac{1}{y^2 - k^2} \leq \sum_{1\leq k\leq E(y)-1} \frac{E(y) - k}{E(y)^2 - k^2} = \sum_{1\leq k\leq E(y)-1} \frac{1}{E(y) + k} \leq 1.$$

and

$$\sum_{E(y)+2\leq k} \frac{1}{k^2-y^2} = \sum_{k=2}^{\infty} \frac{1}{(k+E(y))^2-y^2} \leq \sum_{k=2}^{\infty} \frac{1}{(k-1)^2} = \frac{\pi^2}{6}.$$

Therefore $|\widehat{w}_3(\xi,\lambda)|$ is bounded on the line $\Re(\lambda)=1$.

It remains to find the estimate satisfied by $\widehat{w}_4(\xi,\lambda)$. Indeed, since $(\sqrt{2}\sin(k\pi x))_{k\in\mathbb{N}^*}$ form a Hilbert basis of $L^2(0,1)$, then we may write $\widehat{u}_2, \widehat{w}_2, \widehat{w}_4$ as follows

$$\widehat{u}_2(x,\lambda) = \sum_{k=1}^{+\infty} u_2^{(k)}\sqrt{2}\sin(k\pi x), \widehat{w}_2 = \sum_{k=1}^{+\infty} w_2^{(k)}\sqrt{2}\sin(k\pi x), \widehat{w}_4 = \sum_{k=1}^{+\infty} w_4^{(k)}\sqrt{2}\sin(k\pi x).$$

By the first equation of (3.62), we get

$$\forall k\geq 1, u_2^{(k)} = \frac{w_2^{(k)}}{k^2\pi^2+\lambda^2}.$$

Due to (3.65)

$$\forall k\geq 1, w_4^{(k)} = \frac{\lambda u_2^{(k)}}{ik^2\pi^2+\lambda}.$$

We deduce that

$$w_4^{(k)} = -\frac{\lambda w_2^{(k)}}{(k^2\pi^2+\lambda^2)(ik^2\pi^2+\lambda)}.$$

For $\lambda = 1+iy$ we have

$$|k^2\pi^2+\lambda^2| = \sqrt{4y^2+(1+k^2\pi^2-y^2)^2} \geq 2|y|,$$

and

$$|ik^2\pi^2+\lambda| = |1+ik^2\pi^2+iy| \geq |y|.$$

Hence for $|y|$ large enough we have

$$|w_4^{(k)}| \leq \frac{|w_2^{(k)}|}{|y|}.$$

Using $\widehat{w}_2 = \widehat{w}_3 + \widehat{w}_4$ we get for $|y|$ large enough

$$|w_4^{(k)}| \leq \frac{|w_3^{(k)}|}{|y|}.$$

We finally conclude that for $|y|$ large enough $|\widehat{w}_4(\xi,\lambda)|$ is bounded on the line $\Re(\lambda)=1$. It follows that $|\widehat{w}_2(\xi,\lambda)|$ is bounded as well.

■

In what follows we prove that the observability assumption (O) holds on subspaces of $\mathcal{D}(\mathcal{A}_d)$ on which we deduce the polynomial stability of the energy. Let us first remark that 0 is not an eigenvalue of $\mathcal{A}_d$. Let $\lambda = i\mu$ an eigenvalue of $\mathcal{A}_d$ and $U=(u,v,w)$ a corresponding eigenvector. We then have,

$$\begin{cases} -\mu^2 u - \partial_{xx} u + w = 0 \\ \mu w - \partial_{xx} w - \mu u = i w(\xi)\delta_\xi. \end{cases} \tag{3.67}$$

with

$$u(0) = u(1) = w(0) = w(1) = 0.$$

Multiplying the second equation by $\overline{w}$ then integrating by parts on $(0,1)$, we find that $w(\xi) = 0$. We hence deduce that $w = 0$. Moreover, multiplying the first equation by $\overline{u}$, integrating by parts and considering the imaginary part we deduce that $u = 0$.

In order to verify the observability assumption (O) we study in what follows the spectrum of $\mathcal{A}_c$. Recall that the eigenvalues of $\mathcal{A}_c$ are of the form $\lambda = i\mu, \mu \in \mathbb{R}$.

Proposition 3.6.12 *Let $\sigma(\mathcal{A}_c)$ be the set of eigenvalues of $\mathcal{A}_c$. Then*
(i) Every element of $\sigma(\mathcal{A}_c)$ is simple and $\sigma(\mathcal{A}_c)$ is a disjoint union of three sets:

$$\sigma(\mathcal{A}_c) = \sigma_0 \cup \sigma_1 \cup \sigma_2$$

where σ_0 is a finite set, and there exists $k_0 \in \mathbb{N}^*$ such that $\sigma_1 = \{i\mu_{k,1}\}_{k\in\mathbb{Z},|k|\geq k_0}$, and $\sigma_2 = \{i\mu_{k,2}\}_{k\in\mathbb{N},k\geq k_0}$.

(ii) For $i\mu_{k,i} \in \sigma_i, i = 1,2$, an associated eigenvector $\phi_{\mu_{k,i}} = \dfrac{1}{|k|^{\alpha_i}}(u_{\mu_{k,i}}, v_{\mu_{k,i}}, w_{\mu_{k,i}})$, with $\alpha_1 = 1$ and $\alpha_2 = 4$ is given by

$$u_{\mu_{k,i}}(x) = \sin(k\pi x), v_{\mu_{k,i}}(x) = i\mu_{k,i}u_{\mu_{k,i}}(x), w_{\mu_{k,i}}(x) = (\mu_{k,i}^2 - k^2\pi^2)\sin(k\pi x).$$

(iii) The following estimates hold

$$\mu_{k,1} = k\pi + \frac{1}{2\pi^2 k^2} + o(\frac{1}{k^2}),\ |k| \to \infty, \tag{3.68}$$

$$\|\phi_{\mu_{k,1}}\|_{\mathcal{H}} \sim 1, \tag{3.69}$$

$$\mu_{k,1}^2 - k^2\pi^2 = \frac{1}{k\pi} + o(\frac{1}{k}), \tag{3.70}$$

$$\mu_{k,2} = -k^2\pi^2 + O(\frac{1}{k^2}),\ k \to +\infty, \tag{3.71}$$

$$\|\phi_{\mu_{k,2}}\|_{\mathcal{H}} = O(1), \tag{3.72}$$

$$\mu_{k,2}^2 - k^2\pi^2 = k^4\pi^4 + O(k^2). \tag{3.73}$$

Proof. Let $\lambda = i\mu$ be an eigenvalue of $\mathcal{A}_c$ and $U = (u, \lambda u, w)$ be a corresponding eigenvector of $\mathcal{A}_c$. Then u and w satisfies

$$\begin{cases} -\mu^2 u - \partial_{xx} u + w = 0 \\ \mu w - \partial_{xx} w - \mu u = 0. \end{cases} \tag{3.74}$$

Replacing w in the second equation, we find that

$$\begin{cases} \partial_{xxxx} u + (\mu^2 - \mu)\partial_{xx} u + (\mu - \mu^3) u = 0, \\ u(0) = \partial_{xx} u(0) = u(1) = \partial_{xx} u(1) = 0. \end{cases} \tag{3.75}$$

It is easy to check that $\mu = 0, \mu = 1$ and $\mu = -1$ are not eigenvalues of $\mathcal{A}_c$.

Let $X_1 = \frac{1}{2}(\mu - \mu^2 - \sqrt{\Delta})$ and $X_2 = \frac{1}{2}(\mu - \mu^2 + \sqrt{\Delta})$ be the roots of

$$p(X) = X^2 + (\mu^2 - \mu)X + \mu - \mu^3 = 0,$$

where $\Delta = \mu(\mu - 1)(\mu^2 + 3\mu + 4)$ is the discriminant of p.

Set $t_i = \sqrt{X_i}, i = 1, 2$ then the general form of u satisfying the first equation of (3.75) and the left boundary condition is

$$u(x) = c_1 \sinh(t_1 x) + c_2 \sinh(t_2 x).$$

Considering the right boundary conditions we see that u is non trivial if and only if t_1 and t_2 satisfy

$$\sinh(t_1)\sinh(t_2)(t_1^2 - t_2^2) = 0.$$

But $t_1^2 - t_2^2 \neq 0$, since $\mu \neq 0$ and $\mu \neq 1$. Hence t_1 and t_2 satisfy the following characteristic equation

$$\sinh(t_1)\sinh(t_2) = 0,$$

which gives that $t_1 = ik\pi$ or $t_2 = ik\pi, k \in \mathbb{Z}^*$ i.e $X_1 = -k^2\pi^2$ or $X_2 = -k^2\pi^2$.

Now, we remark that all the eigenvalues of $\mathcal{A}_c$ are simple. Suppose otherwise that there exists a double eigenvalue, then there exist $k_i, \in \mathbb{N}^*, i = 1, 2$ s.t $X_i = -k_i\pi^2, i = 1, 2$. Thus we have

$$\frac{X_1 X_2}{X_1 + X_2} = -\frac{k_1^2 k_2^2 \pi^2}{k_1^2 + k_2^2} = \mu + 1.$$

Now, replacing μ in $X_1 + X_2 = \mu - \mu^2$, we find that

$$2k_1^4 + 4k_1^2k_2^2 + 2k_2^4 - k_1^6\pi^2 - k_2^6\pi^2 + k_1^4k_2^4\pi^4 = 0,$$

which is impossible since π^2 is a transcendental number.

Therefore,

$$u(x) = \sin(k\pi x),\ w(x) = (\mu^2 + X_i)\sin(k\pi x),\ i = 1 \text{ or } 2.$$

Moreover, the eigenvalues of $\mathcal{A}_c$ are formed of two disjoint families of eigenvalues. The first class of eigenvalues is obtained from $X_1 = -k^2\pi^2$, the second class is obtained from $X_2 = -k^2\pi^2$.

Now, we firstly study the asymptotic behaviour of the first class:
since $X_1 = -\mu^2 + \frac{1}{\mu} + o(\frac{1}{\mu}) = -k^2\pi^2$ then $\mu = k\pi + \frac{1}{2\pi^2 k^2} + o(\frac{1}{k^2}), |k| \to \infty$. If we denote by $\{i\mu_{k,1}\}_{k\in Z^*}$ this first class of eigenvalues then the previous estimate is (3.68). Using the previous estimate we directly get (3.69) and (3.70).

Secondly, since $X_2 = \mu + O(\frac{1}{\mu}) = -k^2\pi^2$ we deduce that the large eigenvalues of the second class are negative, and denoting them by $i\mu_{k,2}$ we easily see that (3.71) holds true. Moreover, since $\mu_{k,2}^2 - k^2\pi^2 = O(k^4)$ then (3.72) holds. ■

In order to use generalized Inghams inequalities we need to estimate $\inf\limits_{\mu_{k,1}\in\sigma_1, \mu_{k',2}\in\sigma_2} |\mu_{k,1} - \mu_{k',2}|$. Unfortunately it seems to be a difficult task and it remains an open question. Hence, to get an observability result we will take the initial condition U_0 in some subspaces of $\mathcal{H}$. For this purpose we introduce

$$H_1 = \text{span}(\phi_\mu)_{\mu\in\sigma_0} \cup \text{span}(\phi_\mu)_{\mu\in\sigma_1} \text{ and } H_2 = \text{span}(\phi_\mu)_{\mu\in\sigma_0} \cup \text{span}(\phi_\mu)_{\mu\in\sigma_2}.$$

Before giving an observability result we introduce the set $\mathcal{S}$ of all numbers $\rho \in (0,\pi)$ such that $\frac{\rho}{\pi} \notin \mathbb{Q}$ and if $[0, a_1, ..., a_n, ...]$ is the expansion of $\frac{\rho}{\pi}$ as a continued fraction, then (a_n) is bounded. Recall that if $\pi\xi \in \mathcal{S}$ then

$$|\sin(k\pi\xi)| \gtrsim \frac{1}{|k|},\ k \in \mathbb{Z}^*, \tag{3.76}$$

(see for instance [8]).

Proposition 3.6.13 *1. For all $\xi \in (0,1)$ there are no $T, C > 0$ such that for all $U_0 \in \mathcal{H}$ we have*

$$\int_0^T |w(\xi,t)|^2 dt \geq C_T \|U_0\|^2_{\mathcal{H}}. \tag{3.77}$$

2. Suppose that $\xi \in \mathcal{S}$.
Let $U_0 \in H_1$ and $U = (u, v, w)$ be the corresponding solution of the conservative problem

$$U_t = \mathcal{A}_c U, U(0) = U_0. \tag{3.78}$$

Then there exists T>0 and a constant $c_T > 0$ such that

$$\int_0^T |w(\xi,t)|^2 dt \geq C_T \|U_0\|^2_{\mathcal{D}(\mathcal{A}_c^{-3})}, \tag{3.79}$$

where $\mathcal{D}(\mathcal{A}_c^{-3}) = (\mathcal{D}(\mathcal{A}_c^3))'$, obtained by means of the inner product in X.
For $U_0 \in H_2$ we have

$$\int_0^T |w(\xi,t)|^2 dt \geq C_T \|U_0\|^2_{\mathcal{D}(\mathcal{A}_c^{-\frac{1}{2}})}. \tag{3.80}$$

Proof.

1. Since
$$\lim_{n\to+\infty} \|(i\mu_{n,1} - \mathcal{A}_c)\,\phi_{n,1}\|^2_{\mathcal{H}} + \left\|\begin{pmatrix} 0 & 0 & D^* \end{pmatrix} \phi_{n,1}\right\|^2_U = 0.$$
Which implies according to [36, Theorem 5.1] that we don't have the exact observability, i.e., the inequality (3.77).
2. Let $U_0 \in H_1$. We may write
$$U_0 = \sum_{\mu\in\sigma_0} u_0^\mu \phi_\mu + \sum_{|k|\geq k_0} u_0^{(k)} \phi_{\mu_{k,1}}.$$
Moreover,
$$w(\xi,t) = \frac{1}{|k|}\left(\sum_{\mu\in\sigma_0} u_0^\mu e^{i\mu t} w_\mu(\xi) + \sum_{|k|\geq k_0} u_0^{(k)} e^{i\mu_{k,1}t} w_{\mu_{k,1}}(\xi)\right).$$
Note that $\gamma_1 = \inf_{\mu,\mu'\in\sigma,\mu\neq\mu'} |\mu - \mu'| > 0$, then using Ingham's inequality there exists $T > 2\pi\gamma_1 > 0$ and a constant $c_T > 0$ depending on T such that

$$\int_0^T |w_1(\xi,t)|^2 dt \geq c_T \frac{1}{|k|} \left(\sum_{\mu \in \sigma_0} |u_0^\mu w_\mu(\xi)|^2 + \sum_{|k| \geq k_0} |u_0^{(k)} w_{\mu_{k,1}}(\xi)|^2 \right).$$

Now using (ii), and estimates (3.68),(3.69), (3.70) of Proposition 3.6.12 we get (3.79). For $U_0 \in H_2$, we use analogous argument.

■

Theorem 3.6.14 *1. For any $\xi \in (0,1)$, the system described by (3.58) is not exponentially stable in $\mathcal{H}$.*

2. Let $U_0 \in H_1 \cap \mathcal{D}(\mathcal{A}_d)$, and let U be the solution of the corresponding dissipative problem

$$U_t = \mathcal{A}_d U,\ U(0) = U_0.$$

Then U satisfies,

$$\|U(t)\|^2 \lesssim \frac{1}{(1+t)^{\frac{1}{3}}} \|U_0\|^2_{\mathcal{D}(\mathcal{A}_d)}. \tag{3.81}$$

3. Let $U_0 \in H_2 \cap \mathcal{D}(\mathcal{A}_d)$, and let U be the solution of the corresponding dissipative problem

$$U_t = \mathcal{A}_d U,\ U(0) = U_0.$$

Then U satisfies,

$$\|U(t)\|^2 \lesssim \frac{1}{(1+t)^2} \|U_0\|^2_{\mathcal{D}(\mathcal{A}_d)}. \tag{3.82}$$

Proof.

1. This result is a direct consequence of the first assertion of Proposition 3.6.13 and Theorem 3.4.1.
2. Due to Proposition 3.6.11 and Proposition 3.6.13 we deduce (3.81) from Theroem 3.5.1 setting $\mathcal{H}_1 = \mathcal{D}(\mathcal{A}_c)$ and $\mathcal{H}_2 = \mathcal{D}(\mathcal{A}_c^{-3})$ and $\theta = \frac{1}{4}$.
3. As in 2. we deduce (3.82) setting $\mathcal{H}_1 = \mathcal{D}(\mathcal{A}_c)$ and $\mathcal{H}_2 = \mathcal{D}(\mathcal{A}_c^{-\frac{1}{2}})$ and $\theta = \frac{2}{3}$.

■

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